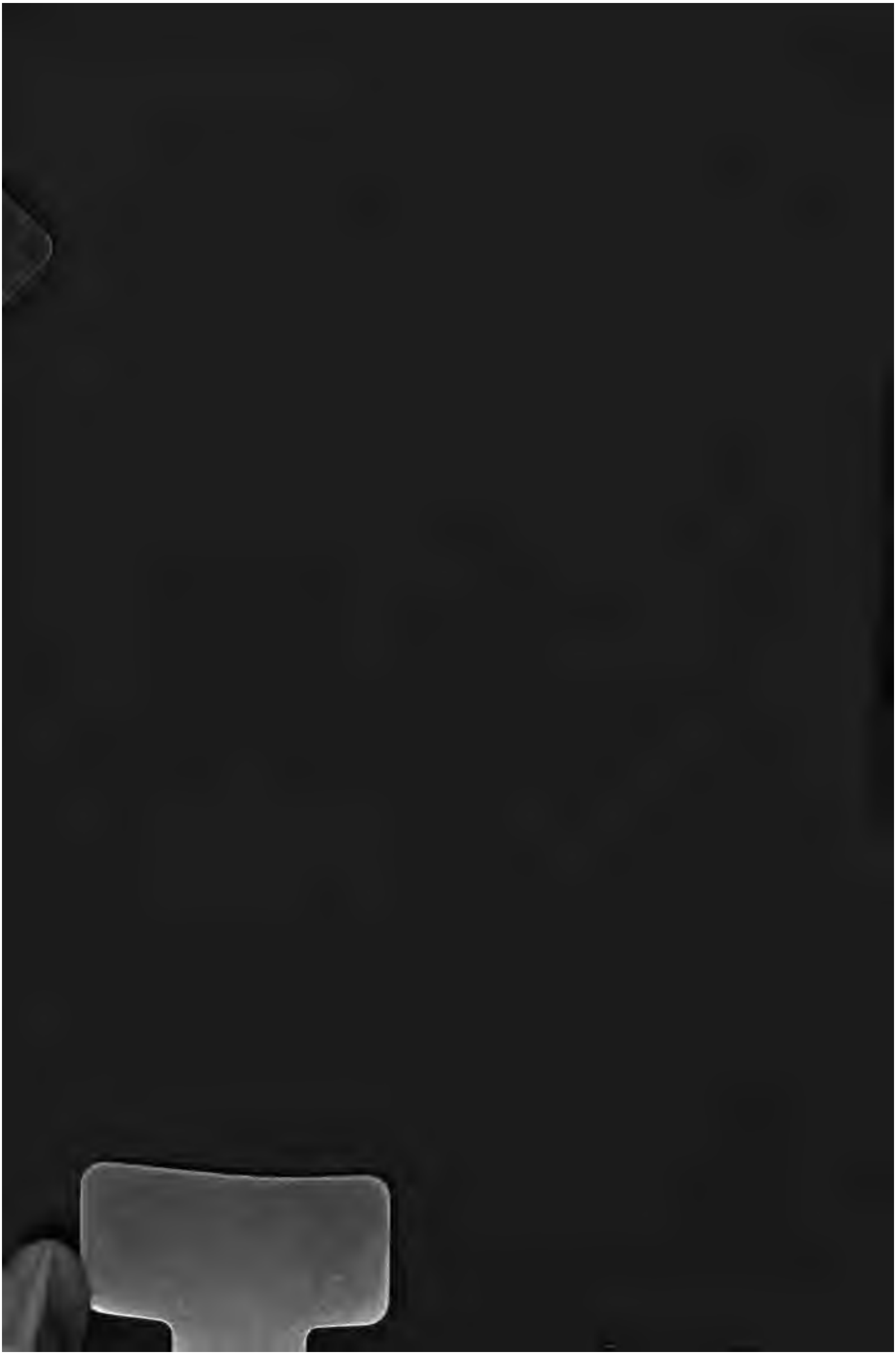

A TEXT-BOOK

OF

ARITHMETIC

FOR USE IN HIGHER CLASS SCHOOLS

BY

THOMAS MUIR, M.A., F.R.S.E.,

MATHEMATICAL MASTER IN THE HIGH SCHOOL OF GLASGOW

LONDON

DALDY, ISBISTER & CO.

56, LUDGATE HILL

1878

LONDON:
PRINTED BY VIRTUE AND CO., LIMITED,
CITY ROAD.

PREFACE.

In writing the present text-book the aim has been to produce a work which should be accurate from the point of view of the mathematician, which should be rational in its mode of treatment as far as this is possible or expedient, which should present the essentials of the subject with the accessories in their proper place as accessories, and which, while suited for the purpose of general mental training, should be equally well adapted as a special preparation for the practical business of life.

Features of detail need not be referred to: an idea of them may be got from an examination of the table of contents, which immediately follows.

The Conclusion, with its notices of books, cannot but be imperfect, both from faults of omission and commission. The attention of teachers who have made a special study of Arithmetic is directed to it, in the hope that they may have the kindness to contribute something to its improvement in the future.

The Exercises have been most carefully worked from the printed book itself, so that the greatest possible accuracy in the Results may be expected. To several gentlemen, and especially to Mr. William Reid, M.A., I am much indebted for help in this direction.

T. M.

HIGH SCHOOL OF GLASGOW,
1st June, 1878.

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ARITHMETIC.

INTRODUCTORY.

1. ARITHMETIC, as the name implies, has *numbers* for its subject-matter. As a science, however, it does not include the whole of this department of knowledge, but only the more elementary portions of it, viz., those in which the numbers symbolized are *known* and *definite*. Moreover, it must be viewed at first more as an *art* than a science; that is to say, while the expounding, so far as possible, of general principles and their bearings is not neglected, still more time and space must be devoted to practical applications of them—to what, in fact, is known as *Computation* or *Calculation*.

2. The fundamental ideas of the science, viz., those of "unity" and "plurality," we begin to acquire on first becoming aware of separate existences of the same kind. It may be that we have, first of all, *concrete* ideas, such as "tree," "man," "house"; next, *composite* ideas, such as "three trees," "three men," "three houses"; thence the *abstract* idea of "three," and, finally, the *general* idea of "number."

3. In the idea expressed by "three trees," "a tree" is called in Arithmetic the UNIT OF REFERENCE, or simply the UNIT. It is that with which the whole idea before the mind is compared. What we view as the unit at one time may at

another be only one of a collection which is taken as the unit, *e.g.*, four plantations, seven forests.

4. The *numbers* are commonly understood to be in order, "one," "two," "three," &c.; the term, however, has also a more extended meaning. "One," "two," "three," &c., arise in connection with the *repetition* of the unit; others, also called numbers, arise in connection with its *partition* or *subdivision*. Thus, in the expression "three-quarters of a tree," we may consider the unit to be still "a tree," and not "a quarter of a tree," in which case we look upon "three quarters" as a number, writing it in consequence as one word, "three-quarters." This extended meaning is that adhered to in Arithmetic, and two classes of numbers are thus recognised—(1) INTEGRAL NUMBERS OF INTEGERS, where the unit remains *whole*; (2) FRACTIONAL NUMBERS OF FRACTIONS, where the unit is *broken*.

5. On acquiring the ideas of the various numbers we represent them in *speech*, and at a later period a necessity arises for the ready expression of them in *writing*; in other words, a *Nomenclature* and *Notation* are required. As the series of numbers is infinite, and as a distinct name and symbol for each would be in the last degree perplexing, ingenious *systems* of nomenclature and notation have been devised, easy of acquirement and eminently serviceable.

6. When these have been learned in whole or in part, our comparisons of groups of existences of the same kind are susceptible of the most perfect exactness. Instead of the vague expressions "a small clump of trees" and "a very small clump," we come to say, when necessary, "a clump of ten trees" and "a clump of five trees." Not only, however, do we compare *discontinuous* magnitudes of this kind by using the naturally-existing unit "a tree"; but we extend the use to the comparison of *continuous* magnitudes, such as the *lengths* of two trees. The unit here may, at an early stage, be also, in a sense, *natural*, *e.g.*, a "thumb's breadth," a "foot's length," an "arm's length"; but the

demand for exactness soon requires definite, legalised, ***arbitrary*** units, like the "inch," "foot," "yard."

7. Adopting a different kind of unit we can further express in numbers the *weights* of the trees: taking still another we may with exactness give their *values;* and so on. Numbers in this way come into play in connection with every kind of magnitude in the universe, as soon as we are sufficiently acquainted with it to be able to fix upon a unit of measurement. Arithmetic and its allied branches of the science of measurable quantity, Mathematics, are thus seen to be of fundamental and far-reaching importance.

THE NOMENCLATURE AND NOTATION OF INTEGRAL NUMBERS.

The Nomenclature.

8. The process of counting a number of objects is learned with more or less completeness in learning to speak; and the fact that there is a regular *method* or *system* followed in the naming of the successive integers is sure sooner or later to strike every one. The perfecting of one's knowledge of this process and system is the first requirement in the study of Arithmetic.

9. Let us proceed, therefore, as if counting one by one a collection of separate things, examining carefully the words we use. We say

ONE, TWO, THREE, FOUR, FIVE, SIX, SEVEN, EIGHT, NINE, TEN.

Thus far there is nothing to note in the names, unless it be the entire absence of family resemblance.

10. Next come

eleven, twelve, thirteen, . . . , nineteen,

the last seven of which are clearly similar in structure, and easily recognised as standing for *three-and-ten, four-and-ten,* &c.; the names thus implying that the numbers they represent are viewed in relation to the number *ten*. The same, however, is also true of the first two of the set, the original forms of which have been only somewhat more disguised. *Eleven* means "left one," and *twelve* is "two left"; left, that is to say, after counting *ten*. These numbers, therefore, beyond ten are consciously or unconsciously looked upon as a group of ten and one, a group of ten and two, a group of ten and three, &c.

The name of the next integer is not "ten-ten," but

twenty,

that is, "two-tens"; and then follow with much simplicity

twenty-one, twenty-two, . . . , twenty-nine.

Instead of "twenty-ten" we have

thirty,

that is, "three-tens," and then as before

thirty-one, thirty-two, . . . , thirty-nine;

and thus without any new feature to note we pass over in succession,

forty, forty-one, . . . , fifty, fifty-one, . . . , sixty, . . . , seventy, . . . , eighty, . . . , ninety, ninety-one, . . . , ninety-nine.

Up to this point, then, we find that integers greater than ten are spoken of as containing so many groups of ten, and so many separate units more.

11. After what we have already seen we are not surprised to find following ninety-nine a new form of word, viz.,

HUNDRED,

and then *hundred and one*, *hundred and two*, and so on to *hundred and ninety-nine.* Having had *twenty* used for "two-tens" we should next expect a word denoting "two-hundreds," but for this we have simply *two hundred.* Similarly after *two hundred and ninety-nine* we have *three hundred*, and so on until we have reached *nine hundred and ninety-nine.* Thus far, therefore, all integers greater than ninety-nine are spoken of as containing so many groups of hundreds, so many groups of tens, and so many separate units.

12. For "ten hundreds" a distinct word is employed, viz.,

THOUSAND,

although it would not be inconvenient to say *ten hundred*, *ten hundred and one*, . . . , *eleven hundred*, *eleven hundred and one*, . . . , and so on to *ninety-nine hundred and ninety-nine;* and, indeed, expressions of this kind are often used. From a *thousand* we proceed through *a thousand and one*, *a thousand and two*, &c., to *a thousand nine-hundred and ninety-nine.* Then come *two thousand*, *two thousand and one*, . . . , *three thousand*, &c. For "ten thousands" no single distinct word, as in the case of "ten tens" and of "ten hundreds," is employed; but on we pass through *ten thousand*, . . . , *eleven thousand*, . . . , *one hundred thousand*, . . . , to *nine hundred and ninety-nine thousand nine hundred and ninety-nine.*

13. For the next integer, which is "a thousand thousands," a special name being necessary, we use

MILLION;

and passing on from this we say *a million and one*, *a million and two*, &c., coming in time to *two million*, . . . , *three million*, . . . , *a thousand million*, . . . , and, finally, *nine hundred and ninety-nine thousand nine hundred and ninety-nine million nine hundred and ninety-nine thousand nine hundred and ninety-nine.*

14. As we should next have "a million millions" a new word is again necessitated and BILLION is employed. Continuing to advance we find TRILLION introduced, not, however, to designate "a billion billions" but "a million billions": and similarly a number of "a million trillions" is called a QUADRILLION, and so on.

15. This system of nomenclature, as may have been observed, is not entirely free from irregularities. In *thirteen*, *fourteen*, &c., the larger portion of the number, contrary to the general rule, is placed last; in *eleven* and *twelve* irregularity is still more marked; and the words *hundred*, *thousand*, *million*, *billion*, *trillion* are introduced in accordance with no fixed principle.

16. Notwithstanding these irregularities, however, the system in regard to *power* or *efficiency* is perfect. The few words it employs are—ONE, TWO, THREE, FOUR, FIVE, SIX, SEVEN, EIGHT, NINE, TEN, HUNDRED, THOUSAND, MILLION, BILLION, and the derivatives, *eleven*, *twelve*, *thirteen*, *fourteen*, *fifteen*, *sixteen*, *seventeen*, *eighteen*, *nineteen*, *twenty*, *thirty*, *forty*, *fifty*, *sixty*, *seventy*, *eighty*, *ninety*. By means of these numbers of which we can form no idea can be named with the utmost brevity and ease.

17. The prominent place which the number *ten* occupies in the system cannot fail to be remarked. When we say "*eight million seven hundred and sixty-four thousand three hundred and fifty-two pounds*," we, as it were, mentally resolve the sum into

> *eight* parcels of a *million* pounds each,
> *seven* of a *hundred thousand* pounds each,
> *six* of *ten thousand* pounds each,
> *four* of a *thousand* pounds each,
> *three* of a *hundred* pounds each,
> *five* of *ten* pounds each,
> *two* of *one* pound each;

each parcel of any group containing exactly *ten* of the

parcels in the group following, and the number of parcels in any group being, as a consequence, always less than *ten*. The number which comes prominently forward in this way in the formation of the system is called the BASE of it, and the name of the system is derived from the name of the base. The English system of nomenclature, therefore, is called the DECIMAL system, its base being *ten*.

18. In most languages the base of the system of nomenclature is the same as in our own; but systems on other bases may be conceived of equal efficiency, and have really been in existence. Again, other languages employing the decimal system may not contain the irregularities existing in our language, and may or may not have irregularities peculiar to themselves.

EXERCISES. SET I.

1. Give the integer immediately preceding each of the following: *Seven hundred, two thousand, eighty-four thousand, seventeen million, a thousand billion.*

2. Give instances of names of numbers in English not in accordance with the decimal system.

3. Explain the system of nomenclature of integral numbers in any foreign language, and point out the irregularities existing in it.

4. What is the base of the system of nomenclature in which appears the number *two gross seven dozen and eleven?*

5. Use the nomenclature of the system mentioned in (4) to count *one gross two dozen and ten.*

THE NOTATION.

19. The system of *notation* or written symbolism which we use is even more perfect than our system of nomenclature. Employing only ten separate symbols we are enabled to write any number however great, numbers of even inconceivable magnitude being written with perfect simplicity.

20. The ten symbols are

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9,$$

and stand for

nothing, one, two, three, four, five, six, seven, eight, nine,

respectively. They are called FIGURES or DIGITS; 0 is called a CIPHER, NOUGHT, or ZERO, while the nine others are spoken of as SIGNIFICANT FIGURES.

21. At first sight it might appear hopeless to try to express all possible numbers by means of ten symbols, every one of which is thus at the outset suddenly appropriated. However, by a little inventive faculty it may be fulfilled in various ways: the particular mode which has been adopted having, in addition to the unlimited power which the others may possess, the further merit of unsurpassable conciseness and simplicity. Thus, if in ignorance of the method in use we were asked to express symbolically the number mentioned in § 17, we might, as a first attempt at contraction, write

8 *million* 7 *hundred-thousand* 6 *ten-thousand* 4 *thousand* 3 *hundred* 5 *ten* 2,

and then as an obvious second step

M	H-TH	T-TH	TH	H	T	
8	7	6	4	3	5	2,

making in this way our system of notation broadly dependent on our system of nomenclature. A small difficulty would arise in trying to express the numbers *eleven*, *twelve*, &c., the names of which deviate slightly from the general law of formation; but knowing the law and the forms which the words would take if subject to it, the difficulty would vanish and we should at once write

$$\overset{\text{T}}{1}\,1,\ \overset{\text{T}}{1}\,2,\ \text{\&c.}$$

And this method of notation might have been found, at an earlier stage in the progress of Arithmetic, far from ineffective. An important simplification of it, however, is possible, and this simplification being made we are brought face to face with the recognised standard system. It is clear that the contractions M, TH, H, T merely serve the purpose of *designating* or *naming* the parcels spoken of in § 17, into

which numbers are divided; the system, in short, being that each figure indicates so many parcels, and the initial letter above it gives the name of the class or order of magnitude to which the parcels belong. The question, therefore, arises—Can these class-names be indicated in any other way? And the answer which has been given is virtually—Indicate rank by position, let the individual farthest to the right be always of the lowest rank, and should any grade of rank be unrepresented let its place be preserved but marked vacant. In agreeing to this we establish what is both socially and mathematically called a *convention*, *i.e.*, a principle adopted not of absolute necessity but as a convenience. In accordance with it the above number is now written

8764352;

and if any other such collection of symbols be given us to decipher with this key we find no difficulty in doing so. For example, in

2403

we know that the number of individuals of the lowest rank is three, that there is none of the next highest rank, *i.e.*, tens, four of the next highest, *i.e.*, hundreds, and two of the next, *i.e.*, thousands; so that 2403 must represent *two thousand four hundred and three.*

22. Our system of notation for integral numbers may, therefore, be shortly explained as follows:—Nothing and numbers under ten are denoted each by a separate symbol, and all higher numbers by a collocation being made in accordance with a convention as to precedence. This convention is to the effect that the figure farthest to the right represents so many *separate individuals* or *simple units*, and that each of the other figures represents so many equal-sized *parcels of simple units*, the parcels whose number is given by any one figure being ten times larger than those whose number is given by the figure immediately to the right.

23. As our system of nomenclature is styled *decimal*, so also is the system of notation. It is, besides, commonly spoken of as ARABIC, because introduced to Europe by the Arabs; but it is now believed to be Indian in origin.

24. It should be noted that just as there may exist efficient systems of nomenclature built like ours but on a different base, so we may have corresponding to each such system a perfectly efficient system of notation founded on the Indo-Arabic principle. Thus, with the base *four*, the names and symbols would be in close analogy to the following—

NOMENCLATURE.	NOTATION.
"one"	1
"two"	2
"three"	3
"four"	10
"four-one"	11
"four-two"	12
"four-three"	13
"two-four"	20
. . . .	. . . .

Here, to express any number whatever, only *four* symbols would be necessary.

25. Various systems of notation based on principles other than that just explained have also been in use. As instances of such may be mentioned the old Greek notation, in which the letters of the alphabet were the symbols, and the cumbrous and ill-devised Roman system, which in most civilised nations continues to exist, along with the Indo-Arabic, but in a very subservient position.

EXERCISES. SET II.

1. What collections of units are specified by the figures of a number which occupy the seventh, twelfth, sixteenth, and twenty-sixth places respectively, counting from the place of simple units?

2. What is the greatest number that can be expressed by six digits, and the least that can be expressed by seven?

3. It has been said that there is not a sextillion particles of the size of ordinary grains of sand in the whole mass of the earth. How many digits would be necessary to express this number?

4. Supposing that our system of notation were *duodecimal*, and that its symbols were 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *τ*, *ε*, how should we write the following numbers?—

Three dozen and eleven.
Nine gross and four.
Three dozen and six gross four dozen and two.
Ten dozen gross and one.

5. Supposing that our notation and nomenclature were *duodecimal*, how should we read 12, 20, 200, 888, 6537, 8000?

26. For acquiring a knowledge of this, as of every other notation, some time must be spent upon two classes of exercises: (1) exercises in reading the notation, (2) exercises in writing it.

I. Reading the Notation.

27. In every case we may proceed as in the following example:

8354602.

Beginning on the right we say, "*two units, no tens, six hundreds, four thousands, five ten-thousands, three hundred-thousands, eight millions*"; and thus having given each symbol its exact equivalent in spoken language we may be said to have accomplished the reading. The only objection is that the translation is a *literal* one. We approach nearer to the recognised mode of speech if we repeat the same phrases in the reverse order, viz., "*eight millions, three hundred-thousands*," &c.; and if, in addition, we make such changes as "*three hundred-thousands, five ten-thousands, four thousands*," into "*three hundred and fifty-four thousand*," we arrive at the standard form, viz., "*eight million three hundred and fifty-four thousand six hundred and two.*

28. In the case, however, of the larger numbers, facility in reading is best attained by attention to special practical rules deduced from consideration of the general method just explained.

I. Numbers of more than three but not more than six digits. Rule :—*Counting from the right mark off three digits by a comma; then, beginning on the left, read the two groups of digits thus formed as if they were independent, adding the word* thousand *to the reading of the first group.* For example,

27931 is separated thus, 27,931,

and read as if it were written

27 THOUSAND 931.

II. Numbers of more than six digits. Rule :—*Counting from the right, mark off six digits, and, if more than six remain, other six; and so on: then, beginning on the left, read each of the groups as if they were independent, adding the word* million *to the reading of the second group counted,* billion *to the reading of the third group counted, and so on.* For example,

784268943816238456279

is separated thus,

784,268943,816238,456279,

and read as if it were written

784 TRILLION 268943 BILLION 816238 MILLION 456279.

EXERCISES. SET III.

Write in words or *read* the numbers indicated as follows :—

1. 946812, 217118, 12204, 10610, 1111.
2. 240024, 12011, 100000, 10001, 110100.
3. 1010100, 20200002, 606000660.
4. 2163814216, 315210712228.
5. 2011200121022210, 12000000104000.
6. 1010001000100000, 1000100000001000.
7. 1000000000000, 50050005500500005.
8. 700000000010002000, 6000000600000006.
9. 6000000000000400000000000000020000.
10. 5417116211410010600712000010001.

II. Writing the Notation.

29. Here there is a general requirement similar to that for reading the notation, viz., the ability to name the various unit collections in descending order from any starting-point. Suppose, for example, we were asked to write in figures the number *four million five hundred and two thousand seven hundred and sixty*. Knowing that after millions, which is the highest unit-collection mentioned, come hundred-thousands, ten-thousands, . . . , and since we see that in the given number there are

of millions,	*four*,
of hundred-thousands,	*five*,
of ten-thousands,	*none*,
of thousands,	*two*,
of hundreds,	*seven*,
of tens,	*six*,
of separate units,	*none*;

we therefore write

4502760,

and this correctly represents the number given.

30. In order, however, that the learner may acquire in the readiest way the necessary facility in translating the larger numbers into the notation, special practical rules may be given, dependent upon the general method implied in what immediately precedes.

I. When the given number is not less than a thousand but under a million. Rule:—*Write in figures the number of thousands mentioned; and following this put three figures to denote the number of hundreds, tens, and single units in the remaining part.* Thus, to express in figures the number *seventy-four thousand and seventeen*, we write

74,017.

II. When the given number is not less than a million. Rule:—*Observing which of the unit-collections, million, billion, trillion,* &c., *is first mentioned, write in figures the number of*

such; following this put six figures to denote the number of the next lowest name whether it is mentioned or not; and so on till the number of millions has been written; then put six figures to denote the part remaining.

Thus, to represent in figures the number *one hundred and fifteen quadrillion two thousand and five trillion twenty-five thousand and sixteen million one hundred thousand and four,* we first write the number of quadrillions, 115; then the number of trillions, 2005, or in six figures, 002005; then the number of billions, 000000, as none are mentioned; then the number of millions, 25016, or 025016; and, finally, 100004. In figures the given number therefore stands

115,002005,000000,025016,100004.

EXERCISES. SET IV.

Express in figures the following numbers:—

1. Ninety-eight thousand four hundred and fifty-seven. Two hundred and forty-three thousand six hundred and forty-eight. One hundred and eleven thousand two hundred and twelve.

2. Two hundred and ten thousand three hundred and seventeen. Ten thousand seven hundred and two. Three hundred thousand and ten.

3. One hundred thousand one hundred. Ten thousand and one. Three hundred thousand.

4. Seventy-two million three hundred and forty-eight thousand five hundred and sixty-four. Three hundred and eighteen million one hundred and sixty-four thousand three hundred and fifty-two.

5. One thousand one hundred and eleven million two hundred and twelve thousand three hundred and ten. One hundred and twelve thousand two hundred and eleven million seven hundred and nineteen thousand two hundred.

6. Three hundred million ten thousand and two. One hundred thousand million and forty.

7. Twenty-five thousand and eight million. Thirty billion four hundred thousand.

8. Eight hundred and nine billion nine hundred and eighty thousand and ninety-eight million nine hundred and eight thousand eight hundred and ninety. Two thousand and twenty billion two hundred and two thousand and two million twenty thousand two hundred.

9. Twenty trillion two hundred million. Two hundred trillion twenty billion.

10. Six thousand quadrillion four hundred thousand billion. Twelve quadrillion ten trillion one hundred billion one thousand million one hundred thousand.

30A. The nomenclature and notation of *fractional* numbers will be fully explained in a subsequent section; in the meantime it is sufficient to know that the fractions *one-fourth*, *one-half*, *three-fourths* are represented in symbols thus :—

$\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$.

THE FUNDAMENTAL OPERATIONS OF CALCULATION.

31. In a very early stage of civilisation a person who owned two hundred and eighty-three head of cattle, and whose neighbour was the possessor of four hundred and seventeen, might wish to ascertain (1) the number of his herd in the event of his obtaining possession of his neighbour's, or (2) how many more cattle his neighbour had than he.

To satisfy his first curiosity he might employ a collection of pebbles or other convenient small objects, and, counting into a separate heap four hundred and seventeen of them to represent his neighbour's stock, he might then appropriate the individuals of this heap, saying as he took them one by one, "two hundred and eighty-four now to me, two hundred and eighty-five, two hundred and eighty-six," &c.; and thus would name the number of the imagined combined herd as he laid hold of the last pebble.

In his second inquiry he would obtain his end by counting from two hundred and eighty-four to four hundred and seventeen, taking to himself a pebble as he named each number, and then counting all the pebbles thus taken.

More complicated numerical questions than these would, of course, soon arise; but, it will be seen that there are none such which could not be ultimately solved by exactly similar work, viz., counting collections of pebbles. Theoretically, therefore, the fundamental operation of Arithmetic is COUNTING, in the simple sense of the word; and the art when followed in this manner, in practice, may appropriately be called "*pebbling*" or *calculation* (*calculus*, a pebble).

32. In the modern methods, on the other hand, pebbles or counters are entirely dispensed with. At the outset of his course the learner uses them to acquire results in reference to *integral numbers less than ten*, such as "five and six make eleven," "four times eight are thirty-two;" but these and other like simple facts are expected to be acquired once for all, and *remembered*. In dealing with all other numbers he expresses them in figures, that is, as we have seen, by means of 0 and the symbols for the integers less than ten; then, restricting himself to the consideration of only one figure of each number at a time, he applies his previously-obtained facts regarding the integers less than ten, and is thus enabled to secure in piece-meal fashion the desired result. As distinguished from "pebbling," modern practical Arithmetic may, therefore, be described as "*figuring*,"* and the essential part which our systems of nomenclature and notation play in it cannot but be apparent.

33. In calculation, as thus conducted, there are four fundamental operations, viz., ADDITION, of which the first question in § 31 is an example; SUBTRACTION, exemplified by the second question; MULTIPLICATION, in the case of integral numbers, a short method of performing Additions of a special kind; and DIVISION, the reverse operation of Multiplication.

34. Instead of writing in words that any one of these operations is to be performed, we may do so more shortly by special symbols. The symbols in common use for this

* The strange word *ciphering* is indeed used in this sense.

purpose are $+$, to indicate Addition; $-$, to indicate Subtraction; $\times$, Multiplication; and $\div$, Division. They may be spoken of as "symbols of operation," as distinguished from the figures 0, 1, 2, 3, &c., which are "symbols of number."

Brackets ( ), { }, [ ], are used in Arithmetic to denote that what they enclose is to be considered as a whole. Thus, $8-3$ indicates the subtraction of 3 from 8, while $8-(3+4)$ expresses that there is to be subtracted from 8 *the number resulting from the addition of* 3 *and* 4; and $(12-8)-(6-2)$ means that *the number resulting from the subtraction of* 2 *from* 6 is to be subtracted from *the number resulting from the subtraction of* 8 *from* 12.

Any intelligible collection of Arithmetical symbols is called a *Symbolic Arithmetical Expression*, or simply an *Arithmetical Expression.* 3045 and $(3+4-2)\times(6+3)$ are examples of such.

The equality of two Arithmetical expressions is indicated by the symbol $=$, called the "symbol of equality," and read "equals" or "is equal to"; for example, $5+6=11$.

ADDITION.

35. ADDITION is the operation of finding a single number which contains exactly as many units as are contained in all by several given numbers.

The numbers given to be added are called the *Items* or *Addends*, and the single resulting number is called the *Sum* or *Total.*

The sign of Addition, $+$, is read "*plus*," *i.e.*, *more.* In writing, its position with regard to the items is that of a link connecting each one with the following; for example,

$$3+4+5+2$$

indicates the addition of 3, 4, 5, and 2; while

$$(3+4)+(5+2)$$

indicates the addition of 3 and 4, the addition of 5 and 2, and then the addition of the two sums thus found.

It is manifest that a change of the order in which the items are taken cannot affect the sum; thus,

$$(3+4)+5=(3+5)+4=(5+4)+3.$$

36. ADDITION OF INTEGRAL NUMBERS. The first requirement is to learn, by continued trial with counters such as the fingers, the sum of any two integers less than 10; for example, 9 and 3 make 12, 7 and 8 make 15, &c. Our system of nomenclature then renders it easy to pass from this to the addition of any integer less than 10, and any other integer whatever; for example, 39 and 3 make 42, 197 and 8 make 205, &c. Lastly, when two or more of the items are greater than 10 we proceed as in the following example.

Example 1. Find the sum of 342, 70846, 57, and 3791.

Taking the numbers of simple units specified in the items, viz., 1, 7, 6, 2, we find their sum to be 16, that is, 1 ten and 6 simple units. Noting down the 6 we add the 1 ten to the numbers of tens specified, viz., 9, 5, 4, 4, and finding the sum to be 23 tens, *i.e.*, 2 hundreds and 3 tens, we note down the 3 and carry forward, as before, the 2 hundreds; and continuing this we arrive at the sum 75036. For convenience the items are so written that the figures specifying the numbers of simple units are in one column, those specifying the number of tens in another, and so on. Thus,

$$\begin{array}{r} 342 \\ 70846 \\ 57 \\ \underline{3791} \\ \text{Sum}=75036 \end{array}$$

Example 2. Perform the operations indicated in the expression $(38+48)+(60+762+3)+25$.

$$\begin{aligned} \text{Given expression} &= 86+825+25 \\ &= 936. \end{aligned}$$

Exercises. Set V.

Find the sum of the numbers in each of the following groups :—

(1.)	(2.)	(3.)	(4.)
8194267	2148	1080464	8163452
2483	73949	9898839	21
18647	213756	535	37
924563	1035248	618	84
99999	12614	13102	273176
648	89789	1296485	408
1435827	90504	24836	1910907

(5.)	(6.)
218638493	374216549
516482783	598765678
917843759	489798697
45394826	858789567
57643	377889966
946	664888399
97	897897879
892876483	258375264
5007145	814937628
94893	492683758
458	937628537

(7.)	(8.)
387294947	576877874
989	939486797
78	799887774
9496	666778887
87897	854545458
365678	798889777
9998889	778978977
789697	867867868
7687	766778896
968	498999997
98998765	784536729

(9.)
57396849
38645276
20463827
50263940
89476385
27305249
89898989
74938657
27655672
35864795
63489276
59449595
77777777
68092389
49569427
81354495
62198537
29876543
82754689
53469278
42085376
58427968
65423858
75639494

10. *Add together* fifty thousand three hundred and eight, two thousand and twenty-five, one hundred and seven thousand and eighty-nine, two hundred thousand and fifty, seventeen thousand six hundred and eighty-five, four thousand and eleven, six hundred and one thousand six hundred and ten.

11. *Add together* one hundred and fifty million five hundred and ten thousand one hundred and five, seventeen million eighty-nine thousand four hundred, seven hundred and twenty-nine million thirty thousand and eight, ten million eight thousand seven hundred and fifty, four hundred and sixteen million fourteen thousand and forty, three million seven hundred thousand nine hundred and twelve, five hundred million five thousand and fifty.

12. *Add together* thirty million seven hundred and sixteen thousand and eight, fifteen hundred and seventeen million thirty thousand six hundred, twelve thousand and seventy, two billion one thousand million

and twenty, eight hundred and ninety thousand million ninety-eight thousand one hundred and twelve, twelve billion eleven thousand million four hundred thousand, four hundred and ninety thousand and ten million four thousand four hundred.

13. Supply the totals in the following statistical table in reference to the religious denominations of the state of Old Würtemberg in 1861:—

Circles.	Evangelical Lutherans.	Roman Catholics.	Other Christians.	Jews.	Total.
Neckar....	456118	36838	1404	3015	
Black Forest	318065	111747	426	1438	
Jagst.......	259043	113114	347	4249	
Danube ..	146588	265358	322	2636	
Total					

14. Find the number of days from Jan. 29th, 1874, to Nov. 17th, 1874, including both days mentioned.

Perform the operations indicated in the following expressions:—

15. $(7068 + 2130 + 29) + 784 + (103 + 3506) + 10308.$

16. $11394 + (2684 + 35 + 7) + (10 + 13 + 281 + 4).$

17. $(23045 + 6138 + 31 + 4) + (2162 + 303) + (13 + 3013 + 20514).$

SUBTRACTION.

37. SUBTRACTION is the operation of finding the number left when a given number has been removed from a greater given number.

The greater of the given numbers is termed the *Minuend*, the less the *Subtrahend*, and the resulting number the *Remainder*. The Remainder may also be appropriately spoken of as the *Difference* of the two given numbers, or the *Excess* of the one over the other.

The sign of Subtraction, —, is read "*minus*," *i.e.*, *less*. In writing, it is preceded by the minuend and followed by the subtrahend; the minuend, however, not necessarily being the *single* number immediately preceding. Thus, in the expression

$$15 - 4 - 3$$

it is not meant that 3 is to be subtracted from 4, but from

the difference between 15 and 4; $15-4-3$ being, in fact, the same as $(15-4)-3$.

It is manifest that the sum of the remainder and the subtrahend must equal the minuend; thus since $8-6=2$, it follows that $6+2=8$.

38. SUBTRACTION OF INTEGRAL NUMBERS. The knowledge acquired in practising Addition enables us soon to learn to tell the difference between *any integer less than* 10 *and any integer less than* 19; for example, $9-5=4$, $17-9=8$.

Having learned this, we are able to find the difference between *any two integers whatever*, by dealing as in Addition first with those figures which in the given numbers indicate simple units, then with those indicating tens, and so on.

Example 1. From 785 take away 362.

Here the larger number consists of 7 hundreds 8 tens and 5 units, and of these we are to remove 3 hundreds 6 tens and 2 units. Now, when the 2 units have been removed from the 5 units, 3 still remain; when the 6 tens have been removed from the 8 tens, 2 tens are left; and on removing the 3 hundreds from the 7 hundreds we have still 4 hundreds over. Altogether, then, there are remaining 3 units 2 tens and 4 hundreds, *i.e.*, 423. The two given numbers and the result are in practice usually written thus:—

$$\begin{array}{r} 785 \\ \underline{362} \\ 423 \end{array}$$

Example 2. Subtract 2875 from 3168.

$$\begin{array}{r} 3168 \\ \underline{2875} \\ 293 \end{array}$$

Here, on coming to subtract the 7 tens, a difficulty arises, there being only 6 tens specified in the minuend. This we get over as follows:—To the 6 tens we add 10 tens, and, now subtracting from these 16 tens the 7 tens, we have 9 tens, which we note down. On account of this addition of

10 tens we must, in the next step, subtract not 8 hundreds but 10 tens more than this, viz., 9 hundreds. Here again, however, there is a deficiency in the minuend, and, as before, we add 10 to the 1 and then subtract the 9; remembering on account of this addition to subtract, in the next step, not 2 thousands but 3 thousands.

Example 3. Perform the operations indicated in the expression

$$127+16-(2+12)-(200-98).$$

$$\begin{aligned}\text{Given expression} &= 143-14-102\\ &= 129-102\\ &= 27.\end{aligned}$$

38A. If a number is to be increased by another, and the sum diminished by a third number, the same result will be reached and often more easily if the first number be diminished when possible by the third, and the remainder be then increased by the second; for example,

$$81+25-79=81-79+25.$$

The truth stated at the end of § 35 may thus be made more general, as follows:—"*Any possible change of the order in which the addends and subtrahends occur in an expression cannot affect the final result.*"

EXERCISES. SET VI.

1. From 7943816 take 2894609
2. From 5001001 take 697309
3. From 1100501 take 944444
4. From 100041673020 1 take 998943290219
5. From 56401000510 take 56391009511
6. From 43100000012345 take 43000000013346
7. From 21386427851 take 19486457862

8. *Take* twenty-nine million five hundred and three thousand and seventy *from* one hundred and twenty million two thousand and fifty.

9. *From* two hundred thousand and three million seven hundred thousand and twenty *take* one hundred and twenty-three million seven hundred and twenty thousand and fifty-one.

10. *From* one billion one million and two *take* one thousand and one million three hundred and seven.

11. *Take* nine hundred billion five hundred and one thousand million two hundred and two thousand and thirty *from* twenty thousand billion five hundred thousand and ten million sixteen hundred.

12. Find (1) the sum of the sum and difference and (2) the difference of the sum and difference of the two numbers, twenty thousand three hundred and four, and two thousand four hundred and thirty.

13. What must be added to the sum of thirty-nine and one hundred and seventy-nine to produce the difference between twenty-seven and a thousand?

14. Complete the following statistical table in reference to agriculture in Ireland:—

Crops.	No. of acres cultivated in the year.		Increase.	Decrease.
	1860.	1861.		
Wheat......	466415	401243		
Oats........	1966304	1999160		
Barley	181099	198955		
Beer and rye..	12734	11582		
Potatoes	1172079	1133504		
Total				

15. In 1871 Manchester and Salford had a population of 504175, Liverpool 493405, Glasgow 477156, Birmingham 343787, Leeds 259212. Make a table showing at a glance the excess of any one of these over each of the others less than it.

Perform the operations indicated in the following expressions:—

16. $38142-7641+3160-3145+724-30$.

17. $38142-(7641-3160+3145)+(724-30)$.

18. $38142-7641+(3160-3145+724)-30$.

19. $(38142-7641)+3160-(3145-724+30)$.

20. $(38142-7641+3160)-3145-724-30$.

21. Indicate in symbols the following expressions:—(1.) The difference between the sum of the three numbers sixty, thirty-five, seventeen, and the sum of twelve and twenty-one. (2.) The excess of the difference between eighty-two and forty-three over the difference between four hundred and thirty and three hundred and ninety-nine.

MULTIPLICATION.

39. MULTIPLICATION is the operation of finding a number which is a given number of times another given number.

The number specifying how many times is called the *Multiplier;* the other given number is called the *Multiplicand;* and both of these are called *Factors* of the resulting number, which itself is denominated the *Product.*

By the product of *more than two* factors is meant the final product got in multiplying the product of the first two by the third, this product by the fourth, and so on until the last has been used as multiplier.

The sign of Multiplication, $\times$, is read "*multiplied by*." In writing, it is preceded by the multiplicand and followed by the multiplier; thus, "ten repeated seven times" or "seven times ten" is written "10×7," and $3 \times 4 \times 5$, or $(3 \times 4) \times 5$ indicates that the product of 3 and 4 is to be multiplied by 5.

40. Multiplication of Integral Numbers. Beginning as usual with numbers less than 10, we seek to know the result of multiplying 4 by 5, 6 by 3, 9 by 4, and so on. To multiply 4 by 5 is to find the number which is 5 times 4, that is, the number which is the sum of 4 and 4 and 4 and 4 and 4. This we know to be 20, consequently we have

$$4 \times 5 = 20.$$

Similarly

$$6 \times 3 = 6 + 6 + 6 = 18,$$
$$9 \times 4 = 9 + 9 + 9 + 9 = 36;$$

and so on. With these results, which are fundamental, the learner must become perfectly familiar; they are usually arranged in tabular form and committed to memory.

When this has been attained the multiplication of any integer by any other is an operation involving little difficulty.

I. When the multiplier is less than 10.

Example. Multiply 31468 by 4.

What we are here asked is to find the sum of 31468, 31468, 31468, and 31468. To do this we arrange the items as before explained, viz., thus,

$$\begin{array}{r} 31468 \\ 31468 \\ 31468 \\ \underline{31468} \end{array}$$

There is no need, however, to say "8 and 8 are 16, 16 and

8 are 24," &c., for we know that four 8's amount to 32, and we at once write 2 under the column of 8's and carry forward the 3. The sum of the four 6's in the next column we also know at once to be 24, which, on the addition of the 3, becomes 27, and we write down 7 and carry forward the 2; and so on. Further, as it is clear that this process can be gone through without having before us, as above, the repeated item written the full number of times, we see that there is no need for more than

$$31468 \times 4 = 125872.$$

EXERCISES. SET VII.

Multiply

1. 543391645 by 2, by 3, by 4.
2. 219837565 by 3, by 4, by 5.
3. 398017432 by 4, by 5, by 6.
4. 345680162 by 5, by 6, by 7.
5. 983195690 by 6, by 7, by 8.
6. 219537485 by 7, by 8, by 9.
7. 145009989 by 8, by 9, by 2.
8. 739685394 by 9, by 2, by 3.

II. When the multiplier is represented by a significant figure followed by one or more zeros.

Example 1. Multiply 25347 by 10, by 100, by 1000.

The relation in which these multipliers stand to the base of our system of notation makes multiplication by them extremely easy. For, if immediately to the right of the figures representing any integral number a zero be placed, the new number thus found contains the same significant figures as the former number, but each figure refers to a collection of units 10 times greater than it did before, so that the new number must be 10 times the former. Similarly, if two zeros be so placed, we get a number which is a 100 times greater; and so on. Thus, it follows that

$$25347 \times 10 = 253470$$
$$25347 \times 100 = 2534700$$
$$25347 \times 1000 = 25347000$$

. . . .

Example 2. Multiply 87 by 20, by 500, by 3000.

Twenty 87's are the same as ten 87's and ten 87's. Now, ten 87's, as we have just seen, amount to 870, so that twenty 87's must amount to twice 870, that is, 1740. Again, five hundred 87's being the same as five times a hundred 87's must amount to five times 8700, that is, 43500.

Similarly, to multiply 87 by 3000 we first annex three zeros to the multiplicand, thus getting 87000, which now we multiply by 3, and find the desired product to be 261000.

The operations here performed may be indicated in symbols as follows :—

$$87 \times 20=(87 \times 10) \times 2=870 \times 2=1740$$
$$87 \times 500=(87 \times 100) \times 5=8700 \times 5=43500$$
$$87 \times 3000=(87 \times 1000) \times 3=87000 \times 3=261000.$$

EXERCISES. SET VIII.

Perform the following multiplications :—

1. 314268×100000.
2. 257000×30000.
3. 7480952×7000.
4. 1812400×80000.
5. $3046 \times 10 \times 20 \times 300$.
6. $9094 \times 20 \times 3000 \times 900$.
7. $21873 \times 300 \times 70000$.
8. $38745 \times 60000 \times 40 \times 80$.
9. $200 \times 200 \times 200 \times 200 \times 200$.
10. $90 \times 80 \times 900 \times 8000 \times 90000$.

III. When the multiplier is any integral number other than those already referred to.

Example 1. Multiply 14289 by 356.

We are here asked to find the number which results when 14289 is repeated 356 times. Now, when 14289 is repeated 6 times there results, as we know,

85734;

when repeated 50 times the result is

714450;

and when repeated 300 times the result is

4286700.

Consequently the sum of these three results must be the

number which arises when the given number is repeated 356 times. Thus, to multiply 14289 by 356 we multiply 14289 by 300, by 50, and by 6, and add the three results. In practice the process is arranged as follows :—

$$
\begin{array}{r}
14289 \\
\underline{356} \\
85734 \\
714450 \\
\underline{4286700} \\
5086884
\end{array}
$$

the zeros at the end of the fourth and fifth lines being usually omitted.

Example 2. Find the product of 3754126 and 3004700.

$$
\begin{array}{rr}
 & 3754126 \\
 & \underline{3004700} \\
 & 2627888200 \\
 & 15016504 \\
 & \underline{11262378} \\
\text{Product} = & 11280022392200
\end{array}
$$

EXERCISES. SET IX.

Perform the following multiplications :—

1. 31781 by 213, and by 321.
2. 49476 by 345, and by 5340.
3. 78325 by 555, and by 616.
4. 78329 by 6226, and by 3131.
5. 842567×100304.
6. 287163×9090900.
7. $374 \times 6060 \times 1300$.
8. $2001 \times 7700 \times 3040$.
9. $2900 \times 790027 \times 72900$.
10. $3004 \times 4300 \times 4030 \times 340$.
11. $99999 \times 88888 \times 77777$.
12. $987654321 \times 123456789$.

41. If we repeat 5 dots 4 times, thus,

• • • • •

• • • • •

• • • • •

• • • • •

and then look at the number of columns and the number of dots in each column we shall see that when 5 is repeated

4 times the result is the same as when 4 is repeated 5 times, or, that

$$5 \times 4 = 4 \times 5.$$

The general truth of which this is an instance must already have struck the learner; it may be expressed by saying that "*Multiplier and Multiplicand may be interchanged without any effect upon the product.*" This, however, is only a case of a still more general truth, viz., that "*the product of any number of factors is not affected by the order in which they are combined in the multiplying;*" thus,

$$\begin{aligned} 3\times 2\times 5\times 7 &= 3\times 7\times 5\times 2 = 7\times 2\times 3\times 5 = \ldots . \\ &= (3\times 2)\times(5\times 7) = 3\times(2\times 5)\times 7 = \ldots . \end{aligned}$$

Use has been made of this in the preceding paragraph in finding the product of 87 and 20, 87 and 500, &c., where instead of

$$87\times(10\times 2), \quad 87\times(100\times 5), \quad \&c.,$$

we substituted

$$(87\times 10)\times 2, \quad (87\times 100)\times 5, \quad \&c.$$

Similarly if asked to multiply by 16, which is 4×4, we may do so by multiplying by 4 and then multiplying again by 4 the product thus obtained; and so we may proceed in the case of every number which can be expressed as the product of two or more factors.

There is another general truth already taken advantage of which deserves explicit mention, viz., that "*the product of any two numbers, one of which is the sum of several items, is the same as the sum of the products formed by taking each of these items along with the other number.*" Thus:

$$(4+3+7)\times 10 = 4\times 10 + 3\times 10 + 7\times 10$$
$$9\times(2+3+4) = 9\times 2 + 9\times 3 + 9\times 4.$$

42. The product of two or more equal factors is called a POWER of the repeated factor, and is specified as the *second*

power if the factor be repeated twice, the *third* power if it be repeated three times, and so on. For example:

25 which $=5\times5$ is called the *second power of* 5,

and 64 which $=4\times4\times4$ „ *third power of* 4.

To represent products like these there is in use a more concise notation; thus:

for 5×5 we write 5^2, which is read "5 *power* 2,"

for $4\times4\times4$ „ 4^3, „ "4 *power* 3,"

and so on; the number in small figures which specifies the power being appropriately called its *Index*.

EXERCISES. SET X.

1. Find the product of the first fifteen integers.

2. What is the tenth power of 2?

3. Multiply the fifth power of four by the fourth power of five.

4. A trader had 315 railway waggons built at £47 each, and twice as many at £45 each. What would the total cost be?

5. An advertiser says he has lost a pocket-book containing 25 hundred-pound notes, 6 fifty-pound notes, and 17 five-pound notes. What sum has he thus altogether lost?

6. A historical work consists of 6 volumes, and there are 512 pages in each volume and 40 lines on each page. How many lines must there be in the whole work?

7. A managing clerk retired after being 27 years in a situation, during 12 of which his salary was £275 a year, and during the rest £25 higher. What income must he thus have drawn in the 27 years?

Perform the operations indicated in the following expressions:—

8. $(15+14+13)\times(6+5+4)$; $15+14+13\times6+5+4$.

9. $15\times14\times(13+6)\times5\times4$; $15\times14\times13+6\times5\times4$.

10. $15\times(14\times13+6)\times5\times4$; $15\times14\times(13+6\times5)\times4$.

11. $32\times(731-639)+(133-97)\times2-(684-597)\times(301-295)$.

12. $996^2+2\times996\times4+4^2$; $997^2+2\times997\times3+3^2$.

13. $998^2+2\times998\times2+2^2$; $999^2+2\times999+1$.

14. $(12+3)^4+(16-2)^3$; $12+3^4+16-2^3$; $(12+3^4+16-2)^3$.

15. $5\times10^6+6\times10^5+7\times10^4+8\times10^3+9\times10^2+9\times10+2$.

16. $8\times10^7+3\times10^6+5\times10^5+7\times10^4+6\times10^2+7\times10+5$.

17. Find the product of the third powers of the first five integers, and the third power of the product of the same integers.

18. Multiply the difference between *three million two hundred* and *fifteen billion one hundred* by the sum of *one thousand and nine* and *ten thousand and ninety*.

19. Multiply the product of *a thousand and ten* and *two hundred and two* by the product of the fourth power of *two* and the third power of *five*.

20. Indicate in symbols the following expressions:—(1) three times the second power of ten, (2) ten multiplied by the sum of the second powers of two and three, (3) the sum of the first four integers multiplied by their product, (4) the product of the sum and difference of seven and two.

DIVISION.

43. DIVISION is the operation of finding a number the product of which and a given number is another given number.

The number given as a product is called the *Dividend;* the number given as a factor is called the *Divisor;* and the co-factor sought, that is, the number resulting from the operation, is called the *Quotient.* Thus, to divide 32 by 4 is to find a number the product of which and 4 is 32. This number, viz. 8, is the quotient, 32 the dividend, and 4 the divisor.

If we look upon 8 as the multiplier and 4 as the multiplicand in obtaining the product 32, the name *quotient* (Latin *quoties*, English *how many times*) is seen to be appropriate, for 8 then specifies a *number of times.* It must not be forgotten, however, that looking, as we may do, upon 4 as the multiplier and 8 as the multiplicand, the name as applied to 8 is not appropriate, because 8 is then the answer, not to the question "How many times does 32 contain 4?" but to the question "What number repeated 4 times produces 32?"

The sign of Division, $\div$, is read "*divided by.*" In writing, it is preceded by the dividend and followed by the divisor, *e.g.*, $32 \div 4 = 8$.

44. DIVISION OF INTEGRAL NUMBERS. The knowledge acquired in practising Multiplication enables us almost from the first to answer with ease any question in Division, where the divisor and quotient are both less than 10. For example, if asked the quotient in the division of 15 by 5 we at once

say 3, because the product of the divisor and 3 is 15. Similarly, $72 \div 8 = 9$, $36 \div 9 = 4$, and so on. Also, with equal ease, we have such results as—

$$150 \div 5 = 30, \quad 1500 \div 5 = 300, \quad 15000 \div 5000 = 3, \text{ \&c.}$$
$$720 \div 80 = 9, \quad 72000 \div 9 = 8000, \text{ \&c.};$$

and, more easily still, the following in which the divisor is a power of 10 :—

$$110 \div 10 = 11, \quad 2800 \div 100 = 28, \quad 300000 \div 1000 = 300, \text{ \&c.}$$

We soon remark, however, that the answer to such questions cannot always be stated so simply. Thus, if asked the quotient in the division of 23 by 4 we are at present only able to say that it cannot be any integral number; for there is no integral number the product of which and 4 is 23. To distinguish such a case as this from those of the kind preceding it, we say that 23 is not *exactly divisible* by 4 ; an integer being said to be *exactly divisible* by another when the quotient likewise is integral. A more definite answer is obtained by seeking the first integer below 23 which is exactly divisible by 4, viz., 20, and reasoning thus :—The quotient of 23 by 4 is the same as the quotient of 20 by 4 together with the quotient of 3 by 4, and therefore is the same as 5 together with the quotient of 3 by 4. Now, the quotient of 3 by 4 we shall afterwards see is the fractional number *three-fourths*, the symbol for which is $\frac{3}{4}$; consequently the quotient of 23 by 4 is $5 + \frac{3}{4}$, or, as this is usually written, $5\frac{3}{4}$. Similarly,

$$17 \div 8 = 16 \div 8 + 1 \div 8 = 2 + \tfrac{1}{8}, \text{ or } 2\tfrac{1}{8}.$$
$$47 \div 9 = 5\tfrac{2}{9}, \quad 47 \div 8 = 5\tfrac{7}{8},$$
$$117 \div 10 = 11\tfrac{7}{10}, \quad 31582 \div 1000 = 31\tfrac{582}{1000}, \text{ \&c.}$$

Exercises. Set XI.

Give the quotients in the following cases of division :—

1. $30 \div 5$, $30 \div 6$, $35 \div 6$, $33 \div 5$.
2. $56 \div 7$, $56 \div 8$, $58 \div 7$, $59 \div 8$.
3. $29 \div 3$, $47 \div 5$, $83 \div 9$, $26 \div 4$.

4. $27 \div 6$, $27 \div 8$, $55 \div 9$, $61 \div 9$.
5. $2700 \div 9$, $270 \div 90$, $64000 \div 8$.
6. $400 \div 8$, $40000 \div 80$, $4000 \div 50$.
7. $1900 \div 100$, $1986 \div 100$, $1907 \div 100$.
8. $407 \div 10$, $407 \div 40$, $17453 \div 10000$.

The general method of procedure in the division of one integral number by another can now be made clear.

Example 1. Divide 234 by 3.

234 is the sum of 210 and 24, and as $210 \div 3 = 70$, and $24 \div 3 = 8$, it follows that $234 \div 3 = 70 + 8$, that is 78.

Instead of the self-explanatory arrangement of dividend, divisor, and quotient,

$$234 \div 3 = 78,$$

the following is considered more convenient in practice,

$$\begin{array}{r} 3\underline{)234} \\ 78 \end{array}$$

Example 2. Divide 39466 by 7.

$$\begin{array}{r} 7\underline{)39466} \\ 5638 \end{array}$$

Here the process amounts to the breaking up of the dividend 39466 into the items 35000, 4200, 210, and 56, the division of each of these by 7, and summing the quotients thus found. The actual work, however, is as follows:—39 (thousands) $\div$ 7 is 5 (thousands) with 4 (thousands) still to be divided. Taking along with these 4 (thousands) the 4 (hundreds) following, we have 44 (hundreds), and this $\div$ 7 is 6 (hundreds) with 2 (hundreds) still to be divided. These 2 (hundreds) with the 6 (tens) following make 26 (tens), and 26 (tens) $\div$ 7 is 3 (tens) with 5 (tens) still to be divided. Lastly, these 5 (tens) and the 6 following make 56, and $56 \div 7$ is 8.

Exercises. Set XII.

Divide

1. 34692564 by 2, by 3, and by 4.
2. 83946285 by 3, by 4, and by 5.
3. 51946320 by 4, by 5, and by 6.
4. 21594325 by 5, by 6, and by 7.

5. 76420001 by 6, by 7, and by 8.
6. 78456394 by 7, by 8, and by 9.
7. 34956213 by 8, by 9, and by 11.
8. 21431621 by 9, by 11, and by 12.
9. 234856 by 12, 294728 by 12.
10. 16451204 by 12, 1180137 by 12.

Speaking generally, we may say that each step of the process just explained becomes more troublesome to the mind of the worker as the divisor becomes larger.

Example 3. Divide 3186 by 127.

$$127\overline{)3186}$$
$$25\tfrac{11}{127}.$$

Our first difficulty here is to know what the quotient of 318 by 127 is. Looking at the figure 1 in the highest place of the divisor, and the figure 3 in the highest place of the dividend, we see that the quotient cannot be greater than 3; and looking at the next figure in both we see that it cannot even be 3, the product of the divisor and 3 being, in fact, 381, which is much greater than the dividend 318. Trying 2, therefore, we find the product of it and the divisor to be 254, and on subtracting this from 318 there is 64 left. We thus discover that 318÷127 is 2 with 64 still to be divided. Passing on to the next step we now inquire what 646÷127 is, and find it in a similar manner to be 5 with 11 still to be divided. The multiplications and subtractions here performed being no longer easy to the unaided mind, but such as to force the learner to have recourse to figuring, it becomes a question how this can be most conveniently arranged; and the arrangement which has been generally adopted is as follows:—

$$\begin{array}{r l}
127)3186(& 25\tfrac{11}{127}=\text{quotient.} \\
\underline{254} & \\
646 & \\
\underline{635} & \\
11 &
\end{array}$$

Example 4. Divide 12153609 by 39, by 392, and by 405.

```
(1.) 39)12153609(311631
        117
        ---
         45
         39
         --
          63
          39
          --
          246
          234
          ---
           120
           117
           ---
             39
             39
             --
```

```
(2.) 392)12153609(31004 41/392
         1176
         ----
          393
          392
          -----
            1609
            1568
            ----
              41
```

```
(3.)
   405)12153609(30008 369/405
       1215
       -----
        3609
        3240
        ----
         369
```

Exercises. Set XIII.

Divide

1. 31684256 by 103, and by 114.
2. 48156004 by 1013, and by 1127.
3. 47834527 by 234, and by 245.
4. 79316821 by 2345, and by 3456.
5. 10000000 by 748, and by 824.
6. 39612945 by 9372, and by 4613.
7. 37682105 by 941, and by 628.
8. 51362747 by 2567, and by 1712.
9. 19456325 by 386, and by 495.
10. 51377284 by 4992, and by 6999.
11. 38462173 by 898, and by 999.
12. 9999800001 by 9999, and by 99.

45. The fundamental principle of the process explained in the preceding paragraph is that "*the division of the sum of several items by any number yields the same result as dividing each of the items by the number and taking the sum of the quotients thus found.*" Thus in the division of 234 by 3 we reasoned as follows:—

$$
\begin{aligned}
234 \div 3 &= (210 + 24) \div 3 \\
&= 210 \div 3 + 24 \div 3 \\
&= \quad 70 \quad + \quad 8 \\
&= \quad 78.
\end{aligned}
$$

And since 234 is also the sum of 150, 69, and 15 we may obtain the same result in another way, viz.,

$$\begin{aligned} 234 \div 3 &= 150 \div 3 + 69 \div 3 + 15 \div 3 \\ &= 50 \quad + 23 \quad + 5 \\ &= 78. \end{aligned}$$

There is, besides this, another general truth which is sometimes serviceable, viz., that "*division by the product of any two factors yields the same result as division by one of them and division of this quotient by the other.*" Thus, to divide 1008 by 28 we may divide by 4 and then divide by 7 the quotient thus obtained; that is, in symbols,

$$1008 \div (4 \times 7) = (1008 \div 4) \div 7.$$

As a consequence of this, we have a similar truth in reference to a product of any number of factors, the order in which they are taken as divisors being immaterial.

46. We have seen that every operation of division may be viewed as giving the answer to two different questions. This distinction is made more apparent when, instead of dealing with abstract numbers, we refer to a concrete unit of measurement such as the *gallon;* for then the questions before stated may take the forms :—

(1.) "How many times could a 4-gallon measure be filled from a cask containing 32 gallons?" the answer to which is "8 *times*."

(2.) "What size of measure will be filled exactly 4 times from a cask containing 32 gallons?" the answer here being "a measure of 8 *gallons*."

Further, it should be noted that in connection with questions of the second kind there is a phraseology employing the terms *half, third part, fourth part* or *quarter*, &c., which is in very common use; thus we ask "What is the tenth part of 40?" "What is the fourth part of 32 gallons?" and so forth.

47. If we multiply by any number, and divide the product

by the same number, the result, of course, must be the original multiplicand.

We thus see that multiplication by 5 is the same as multiplication by 10 (*i.e.*, 5×2) followed by division of the product by 2; and division by 5 the same as division by 10 followed by multiplication of the quotient by 2; and, similarly, the result of multiplying or dividing by 25, 125, &c. (that is, 5^2, 5^3, &c.), may be more easily got than by the ordinary direct method, 25 being $=100 \div 4$, $125=1000 \div 8$, and so on.

If a number is to be multiplied by another, and the product divided by a third number, the same result will be reached if the first number be divided by the third and the quotient be then multiplied by the second: thus $(315 \times 60) \div 63=(315 \div 63) \times 60$. When the first number is exactly divisible by the third, as in this example, the final result is more easily attained by performing the division first; otherwise, by performing the multiplication first, the learner will avoid being troubled by fractional numbers.

Exercises. Set XIV.

1. Divide three million two thousand one hundred by twenty-five.
2. In an operation of division, the dividend was 986013 and the quotient 987. What was the divisor?
3. Find the ninth part of the seventh part of 21369537.
4. How often is the third power of 5 contained in the fourth power of 80?
5. 97 railway waggons cost 4171 pounds. What was the cost per waggon?
6. How many yards of cloth at 17 shillings a yard can be purchased for 1683 shillings?
7. 4680 hundredweight of coal is to be distributed equally among 312 families. How much should each family receive?
8. Find the twenty-ninth part of thirty million ten thousand and one.
9. Divide the ninth part of 733689 by the ninety-ninth part of 34254.
10. From a bag containing £2500 a person withdraws a tenth, then a fifteenth of the remainder, and lastly a twentieth of what is still left. Find how much he has taken in all.

Perform the operations indicated in the following expressions:—

11. $(9997000029999 \div 9999) \div 9999$.

12. $(170+4318-214-40+6)\div(613-215+26)$.

13. $120\times 30\div(6-4+98)$, $120\times(30\div 6)-(4+98)$.

14. $(16^4\div 2^6)\div 4^2$, $(15^3\div 3^3)^2\div(10^6\div 2^6)$.

15. $(97^2+2\times 97\times 3+3^2)\div(91^2+2\times 91\times 9+9^2)$.

16. $16^4\div(2^6\div 4^2)$, $(6^3\div 2^3+8^2\div 4^3-9^3\div 3^6)\div 3^3$.

17. State in symbols the fact that if the fourth power of eight be divided by the sixth power of two the quotient is also the sixth power of two.

18. A person buys 193 oxen for £5404 and 17 score of sheep for £680. How many sheep is one of the oxen worth?

19. Write down the following sentences in symbols:—(1) The difference between the fourth part and the fifth part of a hundred is the twentieth part of a hundred. (2) The third part of the fourth part of twelve is one.

20. Express by means of arithmetical symbols that *the difference between the product of the three numbers four, five, six, and the quotient of three hundred and fifteen by nine is to be multiplied by the sum of twenty-four and the fourth power of two.*

Examination Papers on §§ 1—47.

I.

1. Write in *words* the numbers 20010, 1002011, 200100010.

2. Write in *figures* the numbers *one hundred thousand and forty, fifteen million twenty thousand and eleven, three thousand million three thousand and six.*

3. Explain and illustrate with examples the terms *factor, product, quotient.*

4. Explain fully the process of subtracting 892 from 3001.

5. Divide the product of 3942 and 5876 by the difference between nineteen score and fourteen dozen.

6. A shopman who on starting had 1120 pounds of sugar sold in one week 526 pounds, and in the next 423 pounds, when he again got in a supply of 1120 pounds. How much would he then have in stock?

7. Find a single number equivalent to

$$1095+(318-199)\times(1728\div 36).$$

II.

1. How much does the product of *eleven thousand two hundred* and *one hundred thousand and sixty-nine* exceed the sum of *ten thousand million four hundred and twenty thousand* and *two thousand million four hundred and two.*

2. Find the number which subtracted from 80000 leaves 57735; and divide it by 365.

3. Explain the terms *dividend* and *quotient.* When is an integer said to be *exactly divisible* by another?

4. Find the number which divided by 398 gives the same quotient 20736 divided by 432.

5. Multiply 325 by 713, and explain the process.

6. In one workshop 14060 nails are made in a day, in another 129400. How many nails more than the other must the first shop make in a year of 296 working days?

7. Perform the operations indicated in the expressions

$$31 \times 37 - 29 - (162 - 63 + 9), \quad 31 \times (37 - 29) - (162 - 63) + 9.$$

III.

1. Divide the difference between *one billion forty-four thousand million three hundred* and *one hundred thousand and four million six thousand and one* by the product of *nineteen hundred and two* and *one hundred and ninety-two.*

2. By what must the number whose twenty-fifth part is 639 be multiplied so as to produce 1581525?

3. What is meant by the *fifth power* of any number? Find the fifth power of the second power of 5.

4. Divide 832 by 13, and explain the process.

5. Perform the operations indicated in the expression

$$(16^2 - 12^2) \times 14 + 14 \times (16 + 2^2)^2.$$

6. A watchmaker purchases 15 watches at 17 pounds each, 13 at 15 pounds each, and 26 at 16 pounds each. How much does he gain by selling them all at £20 each?

7. Express in words the general truths of which the following statements are particular instances:—

$$(3 + 7 + 2) \times 5 = 3 \times 5 + 7 \times 5 + 2 \times 5.$$
$$315 \div (5 \times 9) = (315 \div 5) \div 9.$$
$$519 - 327 = (519 + 100) - (327 + 100).$$

UNITS OF MEASUREMENT.

48. Numbers are first used, as we have seen, in connection with natural objects separately existing; for example, six *men*, three *apples;* and then later in the comparison of continuous magnitudes of the same kind, such as the *lengths* or *weights* of objects. This comparison is accomplished by fixing upon a magnitude (*e.g.*, the yard, pound, acre, &c.) of the same kind as the magnitudes to be compared, and

ascertaining how many times it is contained in each of them. In so doing we are said to *measure* the magnitudes, and the fixed magnitude employed is called the *unit* of measurement.

If the unit be not contained an exact number of times in any one of the magnitudes, and there thus be a portion left unmeasured, we may proceed towards the exact measurement of the magnitude by two slightly different modes. One mode, which will be explained afterwards, introduces fractional numbers; the other is, to adopt a *smaller unit* (*e.g.*, the inch, ounce, rood, &c.) to measure the portion remaining, and after this a *still smaller* if there be again a remainder, and so on; so that the measurement comes to be expressed in terms of *more than one unit.* In this way there arises a set of units corresponding to each kind of magnitude which it may be found necessary to measure, one set for length, another for weight, a third for time, and so forth.

49. At an early stage, great exactness would not be demanded in the units employed; the unit of length, for example, might be *the length of a man's foot,* and would thus vary with the individual; but as civilisation and science advanced, greater exactness would be required; the instruments for measuring would be capable of greater accuracy, and laws would be enacted making the units more definite.

Again, the units being arbitrarily chosen, there would arise in different countries, and often even in different districts of the same country, and among different classes of the same community, different sets of units for the measurement of the same kind of magnitude; German units of weight differ from English units of weight, and not long ago there was considerable diversity in these units in England itself. As intercommunication increases this variety in the units is the cause of much inconvenience, which sooner or later must lead to the predominance of one set. It is thus very common to find at one time in a country many units more or

less obsolete, units in actual use, and units proposed to supplant these.

From the first it would be necessary to know how the various units of a set were related in magnitude to each other. Thus, if two units of length were the *foot's length* and the *thumb's breadth* it would need to be known how often the smaller unit was contained in the larger. If it was not contained an exact number of times, the number of times which gave the nearest approximation would be taken, and thus each of the perfectly definite units which in time arose would contain the next smaller unit an exact number of times. The same would necessarily happen when a new unit was obtained by breaking up a previous one into a number of equal parts and taking one of the parts (*e.g.*, *quart*, &c.), or by repeating a previous one a certain number of times (*e.g.*, *hundredweight*, &c.).

The number of times which one unit contains the next smaller of the set is a matter of much importance in regard to the ease and speed of calculation. After a little knowledge of Arithmetic it is easily seen that it is best (1) to have it the same in the case of every pair of units, and (2) to choose for this purpose the number which is the base of the system of numerical nomenclature and notation. In the adoption of units these facts were not taken into account until the difficulty of change had become very great; but in civilised countries they are now very generally recognised, and, as a consequence, *decimal* systems of units may soon be employed throughout the world.

50. In our own country we have in use sets of units which in this respect are nearly as troublesome as could well have been devised. Besides, we have not yet got rid of the inconvenience of having different sets of units for measuring the same kind of magnitude; our grocers use one set of "*weights*," goldsmiths another, and some apothecaries a third. In point of definiteness, however, little is left to be desired in regard to the units necessary for use in the ordi-

nary affairs of life, viz., the units of TIME, SPACE (including *Length*, *Surface*, and *Solidity* or *Capacity*), MASS (or *Weight*), and MONEY; and much is being done to render the units of *Heat*, *Force*, *Electricity*, &c., equally definite and uniform.

The standard unit of Time, in this as in other civilised countries, is the *Mean Solar* DAY. The solar day (that is, the interval between any two successive passages of the sun across the meridian of any place) and solar year are units supplied by nature, and must have been unconsciously and of necessity adopted; but being afterwards found to be slightly variable the *mean* or *average* was fixed upon. Unfortunately the smaller unit is not contained an exact number of times in the larger, the nearest integer being 365.

The standard unit of Length is the *Imperial* YARD. It is fixed by Act of Parliament to be the distance between two points on a metallic bar kept in the office of the Exchequer in London, the bar being during measurement at the temperature of 62 degrees Fahrenheit. From this we derive a unit of Surface, viz., the *Square Yard*, that is, a square each of whose sides is a yard in length; and a unit of Solidity, the *Cubic Yard*, that is, a cube each of whose edges is a yard in length. For special purposes, however, another unit of this latter kind is employed, viz., the *Imperial Gallon*, which also is defined by Act of Parliament.

The standard unit of Mass (or Weight) is the *Imperial* POUND *Avoirdupois*. It is fixed by Act of Parliament to be the mass of a stamped piece of platinum kept in the office of the Exchequer in London.

The standard unit of Money is the POUND *Sterling*, represented by the gold coins known as "*sovereigns.*"

51. The more important details which it is necessary for the learner to know in connection with these subjects will be found collected in the following "Tables of Units of Measurement," in which are given the names of the various units, the contractions most commonly employed for the names, the relation of any one unit to the next higher of its

set, and, following each table, any remarks that may be necessary regarding the more notable units which are limited in use or becoming obsolete.

Tables of Units of Measurement.

Money.

4 *farthings*	= 1 *penny*	(*d.*)
12 pence	= 1 *shilling*	(*s.*)
20 shillings	= 1 POUND	(£.)

In writing, farthings are looked upon as parts of a penny, "one farthing," "two farthings," and "three farthings," being written "¼d.," "½d.," "¾d.," respectively; thus, 6s. 0¼d., 2½d., £5 12s. 11¾d. The *guinea*, an almost obsolete unit equal to 21s., has no coin corresponding to it.

Time.

60 *seconds* (*sec.* or ^{s}.)	= 1 *minute*	(*min.* or ^{m}.)
60 minutes	= 1 *hour*	(*hr.* or ^{h}.)
24 hours	= 1 DAY	(*da.* or ^{d}.)
7 days	= 1 *week*	(*wk.*)

In measuring long intervals there is employed the *mean solar year* of very nearly 365 da. 5 hr. 48 min. 49½ sec.; that is, the average period between two successive passages of the sun through a certain point (the vernal equinox) in his apparent path. Astronomers use the *sidereal day* and *sidereal year*, which differ but slightly from the *mean solar day* and *mean solar year*. The term *year* used, as it often is, without any specifying adjective is an indefinite unit of about 365 days.

Mass (or Weight).

16 *drams* (*dr.*)	= 1 *ounce* (*oz.*)
16 ounces	= 1 POUND (lb.) = 7000 *grains* (*gr.*)
14 pounds	= 1 *stone* (*st.*)
28 pounds	= 1 *quarter* (*qr.*)
4 quarters	= 1 *hundredweight* (*cwt.*) = 112lb.
20 hundredweights	= 1 *ton*.

Besides this set of units, called "Avoirdupois," there is another set called "Troy," used sometimes in weighing gold, silver, jewels, &c., viz., the *pennyweight* (*dwt.*), which = 24 of the grains mentioned above, the *Troy ounce*, which = 20 pennyweights, and the *Troy pound*, which = 12 Troy ounces, or 5760 grains. Besides the *grain* many physicians continue to use in prescriptions the *scruple* (℈) which = 20 grains, and the dram (ʒ) which = 3 scruples; 8 apothecaries' drams being thus = 1 Troy oz.

LENGTH.

12 *inches* (*in.* or ″)	= 1 *foot* (*ft.* or ′)
3 feet	= 1 YARD (*yd.*)
5½ yards	= 1 *pole* (*po.*)
40 poles	= 1 *furlong* (*fur.*) = 220 yd.
8 furlongs	= 1 *mile* (*mi.*) = 1760 yd.

Cloth is measured in *yards* and *quarters* (of a yd.), or sometimes by the *ell*, which in our country is equal to 5 quarters (of a yd.); depth by the *fathom*, which varies, but is usually equal to 6 ft.; the speed of ships by the *knot* or *nautical mile*, which also varies, but in the Royal Navy is equal to 6086½ ft.; and the surveyor, who measures with a chain 22 yards long and containing 100 links, reckons in *chains* and *links*. The *league* of 3 miles is now almost obsolete.

SURFACE.

144 *square inches* (*sq. in.*)	= 1 *square foot* (*sq. ft.*)
9 square feet	= 1 *square yard* (*sq. yd.*)
30¼ square yards	= 1 *square pole* (*sq. po.*)
40 square poles	= 1 *rood* (*ro.*)
4 roods	= 1 *acre* (*ac.*) = 4840 sq. yd.
640 acres	= 1 *square mile* (*sq. mi.*)

The chain of the land-measurer being = 22 yards, a *square chain* must = 484 square yards, and thus 10 square chains or 100,000 square links = 1 acre.

SOLIDITY OF CAPACITY.

1728 *cubic inches* (*cub. in.*)	= 1 *cubic foot* (*cub. ft.*)
27 cubic feet	= 1 *cubic yard* (*cub. yd.*)

The *cubic mile* also may be used in the case of the measurement of very large magnitudes. These four units like several of those of surface arise naturally from the units of length; in ordinary affairs, however, liquids, corn, &c., are measured by another set of units which long ago had their origin quite independently of the units of length. These are set forth in the following table:—

4 *gills*	=	1 *pint* (*pt.*)
2 pints	=	1 *quart* (*qt.*)
4 quarts	=	1 GALLON (*gall.*) = $277\frac{27}{100}$ cub. in.
2 gallons	=	1 *peck* (*pk.*)
4 pecks	=	1 *bushel* (*bus.*)
8 bushels	=	1 *quarter* (*qr.*)

In the measurement of liquids the peck, bushel, and quarter are not used, but various more or less indefinite units such as the *barrel*, *firkin*, *hogshead*, &c., are employed. Coke is sometimes measured by the *sack* which = 3 bushels, and *chaldron* which = 12 sacks.

In connection with the tables of the units of Length, Surface, and Solidity it should be observed that the number of feet in a yard being 3 it follows of necessity that the number of *square* feet in a *square* yard must be 3 times 3, and the number of *cubic* feet in a *cubic* yard $3 \times 3 \times 3$. Thus, if a square yard be taken, it is easily seen to be divisible as in the following diagram:—

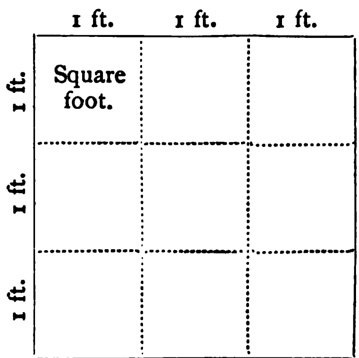

Fig. 1.

Similarly the learner may satisfy himself that the number of square inches in a square foot must necessarily be 12 times 12; and so on.

DIFFERENT WAYS OF EXPRESSING THE SAME MAGNITUDE.

52. On account of the existence of a variety of units for the measurement of each kind of magnitude, it is possible to express any one magnitude in a variety of ways. Thus, there being the inch, foot, yard, &c., for the measurement of length, the same length is expressed by saying "72 inches," or "6 feet," or "2 yards": similarly "50 farthings," "12½ pence," and "1s. 0½d." all express the same sum of money; and "4½ stones," "2 quarters 7 lb.," and "63 lb.," the same mass (or weight).

As it is often of use to be able to express a magnitude in other ways than that in which we find it, it is necessary for the learner to understand how this may be done, and to acquire by practice the requisite facility in doing it.

53. In the first place, then, when a magnitude is given in terms of one unit, and we are required to express it in terms of a lower unit of the same set.

Example 1. Express £16 in *shillings*, and 25s. in *pence*.

(1.) £1 being = 20s., it follows that £16 must = 16 times 20s., that is,

320s.

(2.) 1s. being = 12d., it follows that 25s. must = 25 times 12d., that is,

300d.

It is thus seen that in every case the required number will be found on multiplying the given number by the number of times which the lower unit is contained in the higher.

Example 2. Express £326 in *farthings.*

If we know that £1 = 960 farthings, we may proceed as before, and say that consequently

$$\begin{aligned}\pounds 326 &= 960 \text{ farthings} \times 326\\ &= 312960 \text{ farthings.}\end{aligned}$$

If not, we cannot thus attain the result in one bound, so to speak, but must proceed by easy steps; finding first the number of shillings in £326, then from this the number of pence, and, lastly, the number of farthings. Thus,

$$\begin{aligned}\pounds 326 &= 20\text{s.} \times 326 = 6520\text{s.}\\ &= 12\text{d.} \times 6520 = 78240\text{d.}\\ &= 4 \text{ farthings} \times 78240\\ &= 312960 \text{ farthings.}\end{aligned}$$

In practice this is arranged simply as follows:—

$$\begin{array}{r}\pounds 326\\ 20\\ \hline 6520\text{s.}\\ 12\\ \hline 78240\text{d.}\\ 4\\ \hline 312960 \text{ farthings.}\end{array}$$

Example 3. Express 9 tons in *pounds*, and 16 furlongs in *inches.*

$$\begin{array}{rl}9 & \text{tons}\\ 20 & \\ \hline 180 & \text{cwt.}\\ 4 & \\ \hline 720 & \text{qr.}\\ 28 & \\ \hline 5760 & \\ 1440 & \\ \hline 20160 & \text{lb.}\end{array} \qquad \begin{array}{rl}16 & \text{fur.}\\ 220 & \\ \hline 320 & \\ 32 & \\ \hline 3520 & \text{yd.}\\ 3 & \\ \hline 10560 & \text{ft.}\\ 12 & \\ \hline 126720 & \text{in.}\end{array}$$

Exercises. Set XV.

1. Express 37d. in *farthings*, and £327 in *shillings.*
2. Express 118s. in *pence*, and 250 guineas in *shillings.*

3. Express 24 hr. in *minutes*, and 216 min. in *seconds*.
4. Express 346 qr. in *pounds*, and 3976 tons in *hundredweights*.
5. Express 144 st. in *pounds*, and 2016 lb. in *ounces*.
6. Express 113 mi. in *furlongs*, and 2194 yd. in *feet*.
7. Express 9174 fathoms in *feet*, and 204 chains in *yards*.
8. Express 192 ac. in *roods*, and 17 sq. ft. in *square inches*.
9. Express 28 cub. yd. in *cubic feet*, and 756 cub. ft. in *cubic inches*.
10. Express 1748 gall. in *quarts*, and 6992 qt. in *pints*.
11. Express £298 in *pence*, and 3015s. in *farthings*.
12. Express £714, £860, and £1394 in *pence*.
13. Express 3 da. in *seconds*, and 15 wk. in *hours*.
14. Express 16 tons in *pounds*, and 19 cwt. in *stones*.
15. Express 250 mi. in *poles*, and 17 fur. in *feet*.
16. Express 42 ac. in *square yards*, and 19 sq. yd. in *square inches*.
17. Express 163 cub. yd. in *cubic inches*, and 13 gall. in *gills*.
18. Express 191 qr. and 15 chaldrons in *gallons*.
19. How many fourpenny pieces are equal to 76 crowns, and how many threepenny pieces to 76 guineas?
20. How many halfpence are equal to 94 half-crowns, and how many to 21 half-sovereigns?
21. Find how many inches there are in a mile, how many square inches in a square mile, and how many cubic inches in a cubic mile.
22. Find the difference in grains between 144 lb. Avoirdupois and 175 lb. Troy.
23. Make a table to show at a glance how any one of the four units *farthing*, *penny*, *shilling*, *pound* are related to the others.
24. Do the same for the units of mass.

54. In the second place, when a magnitude is given in terms of more than one unit and we are required to express it in terms of one unit not higher than the lowest unit mentioned.

Example 1. Express £17 16s. 8d. in *farthings*.

Here we might find, as in the preceding exercises, the number of farthings in £17, the number of farthings in 16s., and the number of farthings in 8d., and add the three results together, thus:—

£17 =	17 × 20 × 12 × 4	farthings	=	16320	farthings
16s. =	16 × 12 × 4	„	=	768	„
8d. =	8 × 4	„	=	32	„

and consequently £17 16s. 8d. = 17120 „

But it is more easy to find the number of shillings in £17 and add 16 (thus getting the number of shillings in £17 16s.), then the number of pence in this result and add 8 (thus getting the number of pence in £17 16s. 8d.), and from this the number of farthings. Performing by the mind alone the additions mentioned, we arrange the figuring as follows :—

£17 16s. 8d.
20
356
12
4280
4
17120 farthings.

Here in multiplying by 20 we first add to the 0 of the product the 6 of the 16s. and write down 6, then multiply 7 by 2 and add the 1 of the 16s. ; and similarly in the next step we add the 8, not after all the multiplication by 12 has been performed, but on multiplying the 6.

Example 2. Find (1) the number of *seconds* in 3 wk. 2 da. 7 hr., and (2) the number of *ounces* in 2 tons 18 lb.

3 wk. 2 da. 7 hr.	2 tons 18 lb.
7	20
23	40
24	4
99	160
46	28
559	1298
60	320
33540	4498
60	16
2012400 seconds.	26988
	4498
	71968 ounces.

In finding how many yards are equal to a number of poles, it is necessary to multiply by $5\frac{1}{2}$. For the present let it suffice to say that this is done by taking the *half* of the

multiplicand and *five* times the multiplicand and adding the results, and that similarly we multiply by $30\frac{1}{4}$, &c.

EXERCISES. SET XVI.

1. Express 17s. 2d., £3 19s. 4d., and £2 0s. 6d. in *pence*.

2. Express £3 14s., £16 6s. $0\frac{1}{4}$d., and £3 0s. $0\frac{3}{4}$d. in *farthings*.

3. Express £76 10s. 11d. in *pence*, and £84 0s. $10\frac{3}{4}$d. in *farthings*.

4. Express $7^h\ 43^m\ 27^s$ in *seconds*, 15 da. 41 min. in *minutes*, and 3 wk. 40 min. in *seconds*.

5. Express 4 tons 9 cwt. in *pounds*, 17 cwt. 2 qr. 15 lb. in *ounces*, and 5 tons 1 qr. 13 oz. in *ounces*.

6. Express 17 yd. 10 in. in *inches*, 3 mi. 7 fur. in *yards*, and 4 mi. 412 yd. in *feet*.

7. Express 3 sq. yd. 117 sq. in. in *square inches*, 5 ac. 2 ro. in *square poles*, and 3 ac. 16 sq. po. in *square feet*.

8. Express 9 cub. yd. 17 cub. ft. and 14 cub. yd. 328 cub. in. in *cubic inches*.

9. Express 17 qr. 1 bus. and 41 qr. 1 gall. in *gallons*, and 16 gall. 1 pt. in *gills*.

10. How many fourpenny pieces are equal to £16 18s., and how many to £19 17s. 8d.?

11. How many sixpences are equal to £21 0s. 6d. and how many halfpence to £2 14s. $3\frac{1}{2}$d.?

12. How many stones are equal to 5 tons 13 cwt. 1 qr. 7 lb., and how many halfcrowns to £89 12s. 6d.?

13. How many seconds are there in a mean solar year?

55. In the third place, when the magnitude is given in terms of one unit and we are required to express it in terms of a higher unit of the same set.

Example 1. Express 80s. in *pounds*, and 578d. in *shillings*.

(1) 20s. = £1, and 20s. is contained 4 times in 80s.; consequently 80s. = £4.

(2) 12d. = 1s., and dividing 578 by 12 we see that 12d. is contained 48 times in 576d.; consequently 578d. = 48s. 2d.

It is clear, then, that in every case the number required will be found on dividing the given number by the number of times which the lower unit is contained in the higher.

Example 2. Express 3147 farthings in *pence*, and 786d. in *shillings*.

```
farthings.            d.
4)3147             12)786
  786¾d.              65s. 6d.
```

3147 farthings being thus $=786\frac{3}{4}$d. and 786d. $=65$s. 6d., we see that 3147 farthings $=65$s. $6\frac{3}{4}$d. or £3 5s. $6\frac{3}{4}$d., the usual way of expressing the sum. The figuring necessary to find this last form may, therefore, be arranged as follows:—

```
   farthings.
   4)3147
  12)786    3 far.
   20)65    6d.
      £3    5s.
```

Example 3. Express 37286 lb. in *tons*, &c., and 36725 gall. in *quarters*, &c.

```
   lb.                        gall.
28)37286                    2)36725
  4)1331        18 lb.      4)18362     1 gall.
  20)332         3 qr.       8)4590     2 pk.
       16 tons 12 cwt.         573 qr. 6 bus.
```

The two results thus are
16 tons 12 cwt. 3 qr. 18 lb. and 573 qr. 6 bus. 2 pk. 1 gall.

It is necessary to remark in reference to division by $5\frac{1}{2}$, that to ask how often $5\frac{1}{2}$ yd. are contained in 34 yd. (say) is the same as to ask how often 11 half-yards are contained in 68 half-yards, the answer to which is 6 times and 2 half-yards left over; so that 34 yd. $=6$ po. 2 half-yards., or 6 po. 1 yd. Similarly, instead of asking how often $30\frac{1}{4}$ sq. yd. are contained in 102 sq. yd., we ask how often 121 quarters of a square yard are contained in 408 quarters, and the answer being 3 times with 45 quarters of a square yard left over, we have 102 sq. yd. $=3$ sq. po. 45 quarters of a sq. yd., or 3 sq. po. $11\frac{1}{4}$ sq. yd.

EXERCISES. SET XVII.

1. Express 31728 farthings in *pence*, and 2304d. in *shillings*.

2. Express 299520 farthings in *pence*, 74880d. in *shillings*, and 6240s. in *pounds*.

3. Express 21624 hr. in *days*, and 57600 sec. in *hours*.

4. Express 41440 lb. in *hundredweights*, and 126720 gr. in *Troy pounds*.

5. Express 25344 in. in *yards*, and 5760 po. in *miles*.

6. Express 19360 sq. po. in *acres*, and 45648 sq. in. in *square feet*.

7. Express 2673 cub. ft. in *cubic yards*, and 31808 pt. in *gallons*.

8. Express 3152s., 2104d., and 71073 farthings in *pounds*, &c.

9. Express 11401 farthings, 31846 farthings, and 29001 farthings in the usual form.

10. Express 2183042 sec. in *days*, &c., and 240010 min. in *weeks*, &c.

11. Express 1234 lb. in *hundredweights*, &c., and 37804 oz. in *tons*, &c.

12. Express 9104 st. in *tons*, &c., and 21304 gr. in *Troy pounds*, &c.

13. Express 2761 in. in *yards*, &c., 4604 ft. in *furlongs*, &c., and 4374099 in. in *miles*, &c.

14. Express 2171 sq. po., 31762 sq. yd., and 6394278 sq. in. in *acres*, &c.

15. Express 150314 cub. in. in *cubic yards*, &c., 3141 pt. in *gallons*, &c., and 81723 gall. in *quarters*, &c.

16. How many florins are equal to 7128 threepenny pieces, and how many halfcrowns to 18840 halfpence?

17. Express 30500 guineas in *pounds*, and £39690 in *guineas*.

18. How many Avoirdupois pounds are equal to 18900 lb. Troy, and how many Troy pounds to 518400 lb. Avoirdupois?

56. When two magnitudes of the same kind are expressed in terms of different units it is often not easy to tell at once which of them is the greater. No difficulty should be felt, however, in making such comparisons now that we are able easily to pass from one way of expressing a magnitude to another. Thus, if asked to name the greatest and least of the three magnitudes "113 stones," "15 cwt.," and "16000 oz." the learner would probably at first be quite unable to say; but if he expresses each of them in terms of one and the same unit, the *pound* say, thus:

$$113 \text{ st.} = (113 \times 14) \text{ lb.} \quad = 1582 \text{ lb.}$$
$$15 \text{ cwt.} = (15 \times 112) \text{ lb.} \quad = 1680 \text{ lb.}$$
$$16000 \text{ oz.} = (16000 \div 16) \text{ lb.} = 1000 \text{ lb.}$$

the difficulty has quite vanished.

57. In many of the preceding exercises the labour of calculation is considerable, and in all of them it is much greater than it would be if our units were those of a well-devised system. To see this clearly let us compare the two following exercises:—

(1) Express 73492 lb. in *ounces*.
(2) Express £73492 in *florins*.

There being 16 oz. in a lb., the first number required is got by multiplying by 16, and there being 10 florins in a £ we find the number required in the second case by multiplying by 10. Now every learner knows how very much easier it is to find the latter product than the former, and, consequently, must see to a certain extent how great an advantage it would be to have our units of measurement related to each other after the manner of the pound and florin.

(*Subject continued on p.* 120.)

OPERATIONS WITH MAGNITUDES WHICH ARE EXPRESSED IN TERMS OF MORE THAN ONE UNIT.

58. Another consequence of having units of measurement related to each other as ours unfortunately are, is that additions, subtractions, &c., of magnitudes expressed in terms of more than one unit are often very troublesome; and that, therefore, it becomes necessary for the learner to spend some time in acquiring facility in performing such operations.

59. ADDITION.—Example 1. Find the sum of £2 16s. 4d. and £12 6s. 9d.

The sum of £2 and £12 is £14;
the sum of 16s. and 6s. is 22s.;
and the sum of 4d. and 9d. is 13d.;

so that £2 16s. 4d. and £12 6s. 9d. put together amount to

£14 22s. 13d.

But in a sum of money where shillings are mentioned we very rightly never mention a number of pence amounting to more than a shilling; thus, instead of 22s. 13d. we say 23s. 1d. Similarly, instead of £14 23s. we say £15 3s. Consequently the proper way of expressing the sum found is

£15 3s. 1d.

Example 2. Find the sum of £36 12s. 4½d., £22 13s. 3¾d., 2s. 5¼d., £186 0s. 7½d., and 3¾d.

£	s.	d.
36	12	4½
22	13	3¾
0	2	5¼
186	0	7½
0	0	3¾
245	9	0¾

Here we have placed the items so that the various numbers of pounds are in one column ready for addition, the various numbers of shillings in another, and so on. The sum of the numbers of farthings mentioned we find to be 11, and this being equivalent to 2 pence and 3 farthings, we write down ¾ and add the 2 pence to the other numbers of pence mentioned. The total number of pence we thus find to be 24, which being exactly 2 shillings, we write down 0 under the column of pence and add the 2 shillings to the other numbers of shillings mentioned. The result of this new addition is 29 shillings; consequently we write down 9 under the column of shillings and add 1 pound to the other numbers of pounds mentioned.

Example 3. Add together 6 ac. 3 ro. 36 sq. po. 27 sq. yd., 109 ac. 1 ro. 18½ sq. yd., and 2 ro. 5 sq. po. 24¼ sq. yd.:

ac.	ro.	sq. po.	sq. yd.
6	3	36	27
109	1	0	18¼
0	2	5	24½
116	3	3	9¼

Here we first find the sum of the given numbers of square yards to be 69¾. To ascertain the equivalent of this in poles and square yards, we must seek to know how often 30¼ sq. yd. are contained in 69¾ sq. yd.; or, what is the same thing, how often 121 quarters of a sq. yd. are contained in 279 quarters of a sq. yd. The answer is 2 times with 37 quarters of a sq. yd. remaining; and this remainder being equal to 9¼ sq. yd., we write down 9¼ under the column of sq. yd., and carry on the 2 to the next column.

When the number of items is great it will be found convenient to separate them into groups, then ascertain the sum for each group and add the results.

Exercises, Set XVIII.

Find the sum of each of the following groups:—

1.

£	s.	d.
2	7	6
3	12	2
7	13	7
8	10	10

2.

£	s.	d.
63	17	2
12	18	10
33	14	11
72	16	8

3.

£	s.	d.
75	16	8
0	13	4
92	18	9
47	2	0

4.

£	s.	d.
1	3	4½
3	16	7½
2	14	5
7	3	8½

5.

£	s.	d.
6	18	7½
2	19	4¼
1	2	9½
7	16	2

6.

£	s.	d.
8	12	2¾
2	16	8¼
4	7	2
8	13	4

7.

£	s.	d.
26	3	7½
18	14	9¼
2	0	0¾
19	6	2¼

8.

£	s.	d.
15	2	6
94	12	8¼
94	0	3
72	17	8½

9.

£	s.	d.
33	18	4½
77	19	10¼
85	16	11¼
29	15	8½
37	17	9¼
34	16	7¼

10.

£	s.	d.
85	12	6
0	13	8½
1	16	4¼
72	10	5
94	16	7½
33	7	4¾

11.

£	s.	d.
41	17	8½
92	2	0¼
21	16	7¼
83	13	5
61	12	6¾
27	13	4½

12.

£	s.	d.
94	17	8¾
81	16	7¼
25	13	4¼
73	12	10¼
89	19	11½
67	18	8¼

13.

£	s.	d.
26	12	9
33	15	8¼
76	12	3½
27	16	10¾
0	19	11¼
1	16	3½
39	14	10¼
78	8	4

14.

£	s.	d.
26	16	0¼
91	0	7
48	1	0¾
29	17	10½
33	13	4
27	6	1½
93	0	0¼
0	15	6¾

15.

£	s.	d.
1	2	2
0	17	8¼
0	19	10¾
65	2	8½
3	3	3
0	19	9¼
0	18	11½
6	14	6

16.

£	s.	d.
3184	2	6
17392	12	9
8194	16	5
213	13	10
89784	16	5
7	12	10
9	0	11
8197	17	10
29	6	8
7	16	4

17.

£	s.	d.
9481	13	6
39968	17	10½
26	14	4
39	9	9¾
7297	12	7
96814	13	8
729	6	6½
77	18	9
3814	7	10
918	18	3¾

18.

£	s.	d.
73	13	6
81	12	8
31456	17	5
3	0	0
17	5	6
5	2	9
2187	13	11
9468	12	10
13	18	5
76814	13	8
3184	2	6
468	10	4
759	4	8
1304	12	2
7846	6	9
21825	13	4
62	3	8
139	9	9
45	15	10
276	8	11
7747	12	8
2857	7	4
346	2	10
29	14	9

19.

£	s.	d.
48	12	6½
943	17	8¾
1304	6	9
799	14	6
82	13	11
94	18	10
7688	4	2
8941	12	9
9409	6	8
7998	18	10
4418	12	1½
929	18	4
827	16	8
354	9	9
27	7	7
32	8	9¼
95	19	9
176	18	5
3541	12	2¾
71548	10	4
3946	5	8
219	9	8
35	4	8
7682	12	1

20.

£	s.	d.
99996	7	8
2134	6	5
81947	13	10
1	16	4
3	12	8
16	7	10
39178	17	11
276	18	3
4	12	8
3	0	7
12	13	4
108	18	10
77	12	4
18416	2	8
38893	12	7
72118	13	9
7168	14	11
3419	13	9
79694	17	10
28	2	2
39	1	5
7	16	4
34916	16	4
3473	12	1
2673	13	10
8192	0	0
10	0	0
99416	17	11
13	8	11
299	19	11
134	6	4
27	19	2
31456	12	11
2	6	6
8195	10	10
487	16	6
2994	19	11

21.

£	s.	d.
21	5	8
413	2	6
814	12	9¼
99	13	4
681	16	10
0	15	4¼
3	6	8½
47	3	7
947	13	8
275	12	7
55	14	0½
26	8	11
394	16	2¼
269	12	8½
355	14	2
226	0	0
397	16	8
2	0	10
984	15	1
26	2	3
83	19	4
75	16	8½
29	15	11
384	1	0¼
222	12	2
3	16	3
7	2	11
6	16	10
0	15	8½
89	13	5
926	2	7
333	13	9
27	16	8¼
927	0	0
84	12	6¾
76	18	4
142	7	11¼
284	13	1½
66	9	9
187	18	10
342	12	11¾
67	11	10¼
948	10	1½
27	12	8
345	14	7
728	19	9
37	14	8½
28	10	11¼
341	12	4¾

22.

oz.	dr.
10	13
2	15
14	13
6	10

23.

lb.	oz.	dr.
25	14	2
4	10	12
3	6	10
26	13	13

24.

lb.	oz.	dr.
3	6	7
13	13	14
22	15	11
16	8	13

25.

qr.	lb.	oz.
1	26	12
3	13	14
2	23	6
0	14	14

26.

cwt.	qr.	lb.
3	2	16
10	1	12
16	2	10
4	3	25
16	0	6

27.

cwt.	qr.	lb.
6	0	17
13	1	27
7	2	5
17	3	25
7	1	6

28.

tons	cwt.	qr.
34	2	1
72	11	2
38	17	0
5	14	3
29	2	2

29.

tons	cwt.	qr.	lb.
1	16	1	22
13	18	0	17
76	6	0	12
4	13	2	5
68	17	3	0

30.

hr.	min.	sec.
2	33	20
13	26	12
27	45	45
17	3	36

31.

hr.	min.	sec.
17	2	53
3	39	39
6	47	47
21	50	26

32.

da.	hr.	min.
12	13	41
7	16	37
2	21	24
33	5	16

33.

da.	hr.	min.	sec.
2	13	41	33
1	23	30	36
6	17	43	49
3	21	28	7

34.

wk.	da.	hr.
2	5	17
7	3	12
6	6	16
31	2	23

35.

yr.	da.	hr.
17	216	5
7	320	21
3	275	17
11	96	8

36.

yd.	ft.	in.
12	2	7
7	1	10
16	2	8
9	2	5

37.

yd.	ft.	in.
5	1	11
25	2	7
17	0	5
9	2	2

38.

mi.	fur.	yd.
12	3	24
216	7	118
78	6	219
74	1	79

39.

mi.	fur.	yd.
3	2	116
7	7	78
3	5	155
2	2	213

40.

mi.	fur.	po.
15	6	32
7	2	25
6	3	18
25	7	7

41.

mi.	fur.	po.	yd.
2	3	16	5
2	6	25	3½
5	2	33	1
1	7	7	3

42.

gall.	qt.	pt.
5	3	1
17	2	1
34	2	1
68	1	0

43.

gall.	qt.	pt.
16	2	1½
39	1	1½
7	2	1
95	3	1

44.

qr.	bus.	pk.
19	7	3
26	5	2
18	1	2
39	6	3

45.

qr.	bus.	pk.
55	2	2
73	6	1
53	4	2
7	7	3

46.

bus.	pk.	gall.
5	3	1
16	2	1
31	2	1½
5	1	0

47.

bus.	pk.	gall.	qt.
5	3	1	3
6	1	1	2
5	2	0	1
9	3	1	2

48.

ac.	ro.	po.
16	3	20
33	2	17
18	3	33
7	1	13

49.

ac.	ro.	po.
27	2	35
39	2	28
16	3	37
27	1	35

50.

ac.	ro.	po.
19	0	29
5	1	27
28	3	39
37	3	37

51.

sq. mi.	ac.
397	24
25	468
330	226
75	546

52.

ac.	ro.	po.	sq. yd.
9	2	35	25
1	1	25	16
3	2	7	23
2	3	16	20½

53.

sq. yd.	sq. ft.	sq. in.
5	8	130
2	6	76
2	4	139
3	1	29

54.

sq. yd.	sq. ft.	sq. in.
3	5	68
9	6	79
13	4	126
4	2	34

55.

cub. yd.	cub. ft.	cub. in.
38	17	1000
16	22	685
33	9	729
14	23	1643

56.

cub. yd.	cub. ft.	cub. in.
7	18	216
19	26	495
18	13	1543
75	8	273

57.

cub. yd.	cub. ft.	cub. in.
13	15	1620
26	9	394
18	8	681
29	25	1495

58.

oz.	dwt.	gr.
3	17	21
2	12	7
5	16	23
7	18	17

59.

lb.	oz.	dwt.
5	4	16
6	11	15
3	8	8
4	9	17

60.

lb.	oz.	dwt.	gr.
6	8	19	23
5	7	15	22
9	6	18	19
8	4	9	18

60. SUBTRACTION.—Example 1. From £295 17s. 9¾d. take £147 8s. 2½d.

£	s.	d.
295	17	9¾
147	8	2½
148	9	7¼

Here where the number under each denomination is greater in the first sum than in the second there is no difficulty.

Example 2. Subtract £9 13s. 10½d. from £13 12s. 8¼d.

£	s.	d.
13	12	8¼
9	13	10½
3	18	9¾

In this case the number of farthings in the minuend is not enough to allow of 2 being taken away. Seeing this, we add 4 farthings to the 1 farthing already present, and perform the subtraction. This addition of 1 penny would, if not counteracted, cause an error in the result, but we make it with the intention of undoing it in the next step by

taking away 11d. instead of 10d. Again, however, the number of pence in the minuend is not enough to allow of 11 being taken away, and as before we add to the 8d. already present 12d. more, thus making 20d., from which on taking the 11d. there remains 9d. To counterbalance the effect of the addition of these 12d. we now proceed to take away 14 shillings instead of 13: and so on.

We may view this mode of performing the subtraction in another light, viz., as making repeated use of the principle that the difference between any two quantities remains unaltered if both quantities receive the same increase. We are asked to find the difference between £13 12s. 8¼d. and £9 13s. 10½d., and adding to the one 20s. 12d. 4f., and to the other £1 1s. 1d., we seek instead the difference between £13 32s. 20¼d. and £10 14s. 11½d., which is easily found, and is known to be the same as that sought.

Example 3. From 7 fur. 13 po. 2 yd. take away 3 fur. 25 po. 4½ yd.

fur.	po.	yd.
7	13	2
3	25	4½
3	27	3

The practical value of what is stated in § 38A is easily seen in connection with these examples of subtraction. Thus in Example 2 above we performed the operations which are indicated as follows:—

12d. + 8d. − 11d.
20s. + 12s. − 14s.

whereas if we change the order of the operations, thus,

12d. − 11d. + 8d.
20s. − 14s. + 12s.

the result in each case is more easily got.

Exercises. Set XIX.

	£	s.	d.		£	s.	d.		£	s.	d.
1. From	172	15	8	2. From	374	16	11¼	3. From	76	8	6
take	89	6	3	take	195	11	5¼	take	35	15	3
4. From	381	17	5	5. From	199	2	1	6. From	216	4	4
take	17	16	8	take	74	12	10	take	138	15	8

	£	s.	d.
7. From	13	12	4¼
take	7	13	9½

	£	s.	d.
8. From	128	3	6¼
take	74	11	7¾

	£	s.	d.
9. From	397	17	5¼
take	198	18	11¾

	£	s.	d.
10. From	317	16	11
take	198	18	11½

	£	s.	d.
11. From	211	19	11¼
take	194	19	11¾

	£	s.	d.
12. From	312	13	8
take	194	17	4¾

	lb.	oz.	dr.
13. From	13	12	2
take	6	13	8

	qr.	lb.	oz.
14. From	5	20	12
take	3	22	5

	cwt.	qr.	lb.
15. From	16	1	5
take	8	3	12

	cwt.	qr.	lb.
16. From	25	2	2
take	18	2	11

	tons	cwt.	qr.
17. From	35	17	1
take	18	18	2

	tons	cwt.	qr.	lb.
18. From	2	4	3	16
take	1	16	1	25

	wk.	da.	hr.
19. From	5	6	11
take	2	3	17

	da.	hr.	min.
20. From	11	13	20
take	5	16	30

	hr.	min.	sec.
21. From	23	25	36
take	17	29	50

	hr.	min.	sec.
22. From	6	13	13
take	1	25	46

	yr.	da.	hr.
23. From	5	112	14
take	2	226	17

	da.	hr.	min.	sec.
24. From	2	12	5	20
take	1	16	12	45

	yd.	ft.	in.
25. From	6	2	10
take	3	2	11

	yd.	ft.	in.
26. From	17	1	5
take	8	2	10

	mi.	fur.	yd.
27. From	10	5	111
take	6	7	210

	mi.	fur.	yd.
28. From	16	2	11
take	9	5	89

	mi.	fur.	po.
29. From	23	2	36
take	17	5	17

	mi.	fur.	po.	yd.
30. From	3	6	25	3
take	1	6	32	4

	gall.	qt.	pt.
31. From	27	2	1
take	18	3	1½

	gall.	qt.	pt.
32. From	45	1	0
take	29	2	1½

	qr.	bus.	pk.
33. From	46	3	2
take	27	5	3

	qr.	bus.	pk.
34. From	64	3	1
take	37	7	2

	bus.	pk.	gall.
35. From	34	2	1
take	17	3	0½

	bus.	pk.	gall.	qt.
36. From	17	1	1	2
take	8	1	1	3

	ac.	ro.	po.
37. From	37	1	32
take	18	2	17

	ac.	ro.	po.
38. From	29	0	16
take	19	3	26

	ac.	ro.	po.
39. From	314	1	11
take	157	2	35

	sq. yd.	sq. ft.	sq. in.
40. From	174	8	17
take	39	8	100

	sq. yd.	sq. ft.	sq. in.
41. From	273	1	164
take	189	5	260

	ac.	ro.	po.	sq. yd.
42. From	7	2	14	17
take	1	2	31	25

	cub. yd.	cub. ft.	cub. in.
43. From	390	20	27
take	147	22	123

	cub. yd.	cub. ft.	cub. in.
44. From	179	2	219
take	37	18	884

	cub. yd.	cub. ft.	cub. in.
45. From	741	3	1154
take	394	12	1265

	oz.	dwt.	gr.
46. From	26	15	15
take	13	16	22

	lb.	oz.	dwt.	gr.
47. From	3	10	14	3
take	1	10	5	19

	Troy oz.	gr.
48. From	7	217
take	2	453

49. A man's yearly income is £600, and his expenditure £467 13s. 2½d. What is the excess of his expenditure over his saving?

50. A railway 10 miles long was to be constructed by a squad of men beginning at the one terminus, and a squad beginning at the other. At the end of the first year they had finished 3 miles 260 yd. and 3 miles 157 yd. respectively. How much shorter is the unfinished part than the finished part, and how much longer than the separate portions of the latter?

61. MULTIPLICATION.—Here we have more than one method in use for obtaining the desired result, and though any one is sufficient it is instructive to practise all of them.

I. The multiplier is dealt with in a sense as a whole.

Example 1. Multiply £193 12s. 8½d. by 5.

£	s.	d.
193	12	8½
		5
968	3	6½

Here the process is :—

2f. × 5 = 10f. = 2½d.

½d. put down and 2d. carried on.

8d. × 5 = 40d.; this and 2d. = 42d. = 3s. 6d.

6d. put down and 3s. carried on.

12s. × 5 = 60s.; this and 3s. = 63s. = £3 3s.

3s. put down and £3 carried on.

£3 × 5 = £15; this and £3 = £18; and so on.

With this may be compared the process of finding the sum of 5 items, each of which is £193 12s. 8½d.

EXERCISES. SET XX.

1.	Multiply	£13	16	3	by 2, and by 3.
2.	"	£27	18	4½	by 3, and by 4.
3.	"	£164	18	6¾	by 4, and by 5.
4.	"	£175	16	8½	by 5, and by 6.
5.	"	£94	12	3½	by 6, and by 7.
6.	"	£218	12	2½	by 7, and by 8.
7.	"	£87	16	0¾	by 8, and by 9.
8.	"	£217	16	1¾	by 9, and by 10.
9.	"	£314	7	6¼	by 10, and by 7.
10.	"	£397	10	4½	by 10, and by 8.

11. Multiply £764 18 3¾ by 10, and by 9.
12. „ £397 16 7¾ by 10, and by 11.
13. „ £94 3 7½ by 11, and by 12.
14. „ £68 3 9¼ by 12, and by 10.
15. „ 3 tons 14 cwt. 1 qr. by 6, and by 7.
16. „ 5 cwt. 2 qr. 13 lb. by 5, and by 9.
17. „ 16 yd. 2 ft. 7 in. by 8, and by 10.
18. „ 12 ac. 3 ro. 27 po. by 11, and by 10.
19. „ 3 gall. 2 qt. 1 pt. by 12, and by 20.
20. „ 19 sq. yd. 8 sq. ft. 134 sq. in. by 7, and by 30.

When the multiplier is small, as in the preceding examples, no part of the *process* of calculation need be written down. The mind without aid of this kind can do the required work, and the *results* only are taken note of. When, however, the multiplier is large, the writing out of the process is a great and more or less needful assistance, the best mode of arranging it being as follows:—

Example 2. Multiply £193 12s. 8½d. by 5.

```
      5
      2
    ---
  4)10
    ---
      2   ½
     40
    ---
 12)42
    ---
      3   6
     60
    ---
 20)63
    ---
      3   3
    965
    ---
    968
```

Result,—£193 12s. 8½d. × 5 = £968 3s. 6½d.

This is the example already dealt with at the beginning of the paragraph, and has been reproduced in order that the learner may the more easily see, by comparison with the explanatory process there given, that we have in the above merely arranged in order on paper that which before would

pass through the worker's mind without being taken note of.

Example 3. Multiply 215 ac. 3 ro. 17 po. by 341.

```
       341
        17
      ----
      2387
      341
      ----
  40)5797
      ----
       144   37
      1023
      ----
   4)1167
      ----
       291    3
      1705
      341
     682
     -----
     73606 ac.
```

Result,—215 ac. 3 ro. 17 po. × 341 = 73606 ac. 3 ro. 37 po.

EXERCISES. SET XXI.

1. Multiply	£6	18	4 by 43.		2. Multiply	£12	3	4½ by 143.
3. ,,	£2	15	5¼ by 313.		4. ,,	£7	16	8¾ by 285.
5. ,,	£3	6	5¼ by 295.		6. ,,	£31	18	11½ by 117.
7. ,,	£7	12	8¾ by 741.		8. ,,	£6	16	10 by 377.
9. ,,	£2	17	11 by 839.		10. ,,	£5	15	3½ by 994.
11. ,,	£13	4	7½ by 1847.		12. ,,	£7	13	0¾ by 317.
13. ,,	£6	0	0¾ by 2347.		14. ,,	£0	15	6½ by 773.
15. ,,	£0	0	6¾ by 21304.		16. ,,	£0	9	5 by 3197.

17. Multiply 1 qr. 14 lb. 3 oz. by 207, and by 336.
18. ,, 2 ac. 16 sq. po. by 305, and by 194.
19. ,, 1 hr. 13 min. 43 sec. by 423, and by 579.
20. ,, 3 mi. 117 yd. by 4356, and by 2099.

II. The multiplier is viewed as, in a sense, a composite structure, the result of operations with small integers; thus, 36 may be looked upon as 6×6, and 43 as $6 \times 7 + 1$.

(1.) When the multiplier is a product of small integers; *e.g.*, 160, which $= 10 \times 4 \times 4$.

Example 1. Multiply £5 3s. 2¼d. by 160.

£	s.	d.	
5	3	2¼	
		10	
51	11	10½	=given sum × 10.
		4	
206	7	6	=given sum × 40.
		4	
825	10	0	=given sum × 160.

The foundation for the mode of procedure here is the fact that $160 = 10 \times 4 \times 4$. Similarly, since $160 = 5 \times 4 \times 8$, we may proceed also as follows :—

£	s.	d.
5	3	2¼
		5
25	15	11¼
		4
103	3	9
		8
825	10	0

Exercises. Set XXII.

1. Multiply £1 3 4½ by 16.
2. Multiply £3 12 5¼ by 20.
3. „ £2 13 8½ by 24.
4. „ £7 10 3½ by 35.
5. „ £7 6 10¼ by 72.
6. „ £12 13 1¾ by 84.
7. „ £3 18 7¾ by 100.
8. „ £1 19 0¾ by 1000.
9. „ £6 7 11½ by 320.
10. „ £7 17 7 by 2100.
11. Multiply 1 ton 14 cwt. 3 qr. by 88, and by 96.
12. „ 5 yd. 1 ft. 7 in. by 121, and by 144.
13. „ 3 hr. 17 min. 27 sec. by 1760.
14. „ 1 qr. 7 bus. 3 pk. by 1728.
15. „ 3 ac. 13 sq. po. by 2250, and by 1320.
16. „ 2 cub. yd. 1400 cub. in. by 1280.

(2.) When the multiplier is not a product of small integers, but there is a number near it which is; *e.g.*, 167, which is only 7 more than 160.

Example 2. Multiply £5 3s. 2¼d. by 167.

£	s.	d.	
5	3	2¼	
		10	
51	11	10½	
		4	
206	7	6	
		4	
825	10	0	= given sum × 160
36	2	3¾	= given sum × 7
861	12	3¾	= given sum × 167.

The explanation of this process starts with the fact that

$$167 = 10 \times 4 \times 4 + 7.$$

We first find 160 times the given sum exactly as in the example preceding the last set of exercises, then 7 times, and then add the two results.

Exercises. Set XXIII.

1. Multiply £3 12 7¼ by 29 (which $= 4 \times 7 + 1$).
2. " £7 5 6¾ by 183 (which $= 9 \times 5 \times 4 + 3$).
3. " £3 4 7½ by 211 (which $= 3 \times 7 \times 10 + 1$).
4. " £5 16 10½ by 17.
5. " £1 12 4 by 26.
6. " £2 12 5¼ by 82.
7. " £3 0 6½ by 103.
8. " 5 da. 13 hr. 42 min. by 145.
9. " 2 oz. 12 dwt. 13 gr. by 186.
10. " 3 ac. 3 ro. 17 sq. po. by 122.
11. " 6 cub. ft. 1214 cub. in. by 103.
12. " 4 cwt. 23 lb. by 134.

Although there are usually various ways in which a large multiplier may be viewed as a composite structure, the result of operations upon small integers, still it will generally be found convenient to follow that which is at the basis of our system of numerical notation. Thus, if 327 be the multiplier, we look upon it as $3 \times 100 + 2 \times 10 + 7$, and find first 300 times the multiplicand, then 20 times, then 7 times, and add these three results.

Example 3. Multiply £2 1s. $3\frac{1}{2}$d. by 34126.

	£	s.	d.		
	2	1	$3\frac{1}{2}$		
			10		
	20	12	11	= given sum	× 10.
			10		
	206	9	2	= „	× 100.
			10		
	2064	11	8	= „	× 1000.
			10		
	20645	16	8	= „	× 10000.
			3		
	61937	10	0	= „	× 30000.
Sum in 7th line × 4 =	8258	6	8	= „	× 4000.
Sum in 5th line =	206	9	2	= „	× 100.
Sum in 3rd line × 2 =	41	5	10	= „	× 20.
Sum in 1st line × 6 =	12	7	9	= „	× 6.
Hence, by add$^{n.}$	70455	19	5	= „	× 34126.

Here $34126 = 30000 + 4000 + 100 + 20 + 6$,

$$\text{or} = 3 \times 10000 + 4 \times 1000 + 100 + 2 \times 10 + 6;$$

and this fact forms the basis of an explanation of the mode of procedure.

EXERCISES. SET XXIV.

1. Multiply £2 13 $4\frac{1}{2}$ by 314.	2. Multiply £1 2 $10\frac{1}{4}$ by 223.
3. „ £1 12 $5\frac{1}{4}$ by 1314.	4. „ £3 6 $2\frac{1}{2}$ by 3142.
5. „ £5 0 $2\frac{1}{2}$ by 2715.	6. „ £7 16 $10\frac{1}{4}$ by 1057.
7. „ £6 16 $11\frac{1}{4}$ by 3896.	8. „ £12 3 $5\frac{1}{4}$ by 2003.

9. Multiply 2 qr. 3 bus. 1 pk. by 1030.

10. „ 1 yd. 2ft. $5\frac{1}{2}$ in. by 20430.

III. A third method of obtaining products of this kind is explained in § 63.

62. DIVISION.—It has been already pointed out (§ 46) that when no concrete unit of measurement is referred to, every operation of division gives the answer to two quite distinct questions. When, however, as here, units of

measurement are specified, no such ambiguity is possible. Thus the result of the operation

$$£968\ 13s.\ 6\tfrac{1}{2}d. \div 5$$

can only be viewed as the answer to one question, viz., What sum of money repeated 5 times amounts to £968 13s. 6½d.? and the operation

$$£968\ 13s.\ 6\tfrac{1}{2}d. \div £193\ 14s.\ 8\tfrac{1}{2}d.$$

can arise only from the question, How many times is £193 14s. 8½d. contained in £968 13s. 6½d.?

These two kinds of questions it is necessary to consider separately.

I. When reference is made to a concrete unit of measurement in the dividend only.

Example 1. Divide £968 13s. 6½d. by 5.

$$\begin{array}{r|rrr} & £ & s. & d. \\ 5) & 968 & 13 & 6\tfrac{1}{2} \\ \hline & 193 & 14 & 8\tfrac{1}{2} \end{array}$$

The process here is as follows:—£968÷5=£193 with £3 still to be divided. These £3=60s., and the 13s. specified in the dividend being added, there results 73s., dividing which by 5 we have 14s. with 3s. still to be divided. Expressing these 3s. in pence, and adding the 6d. specified in the dividend, we have 42d.÷5=8d. with 2d. still remaining to be divided. Lastly the 2d. are expressed in farthings, the 2 farthings (½d.) of the dividend are added, and division again performed.

Example 2. Divide £273 6s. 0¾d. by 7, and by 8.

$$\begin{array}{r|rrr} & £ & s. & d. \\ 7) & 273 & 6 & 0\tfrac{3}{4} \\ \hline & 39 & 0 & 10\tfrac{1}{4}\ \tfrac{4}{7} \end{array} \qquad \begin{array}{r|rrr} & £ & s. & d. \\ 8) & 273 & 6 & 0\tfrac{3}{4} \\ \hline & 34 & 3 & 3\tfrac{0}{4}\ \tfrac{3}{8} \end{array}$$

EXERCISES. SET XXV.

1. Divide £98 3 6 by 2, and by 3.
2. " £417 14 6 by 3, and by 4.
3. " £635 18 5¼ by 4, and by 5.

4. Divide £766 5 1½ by 5, and by 6.
5. „ £314 0 0¾ by 6, and by 7.
6. „ £205 7 10 by 7, and by 8.
7. „ £159 13 10¾ by 8, and by 9.
8. „ £696 18 9 by 9, and by 10.
9. „ £140 0 2½ by 10, and by 11.
10. „ £9 0 0 by 11, and by 12.
11. „ 290 hr. 12 min. 5 sec. by 5, and by 11.
12. „ 4 tons 9 cwt. 1 qr. 21 lb. by 9, and by 10.
13. „ 28 yd. 8 in. by 8, and by 12.
14. „ 20 ac. 3 ro. 35 sq. po. by 11, and by 12.

When the divisor is small, as in the preceding examples, no part of the *process* of calculation need be written down, but merely the result. When, however, the divisor is large, the process becomes troublesome to the unassisted mind, and the writing out of the steps is resorted to, the usual mode of arrangement being as follows :—

Example 3. Divide £968 3s. 6½d. by 5.

```
     £    s.  d.    £    s.  d.
5 )968    3   6½( 193   12   8½
   5
   --
   46
   45
   --
    18
    15
    --
     3
    20
    --
    63
    60
    --
     3
    12
    --
    42
    40
    --
     2
     4
    --
    10
    10
    --
```

The divisor here is not large, indeed the example is the same as Example 1, preceding the last set of exercises; the object in repeating it being to enable the learner easily to see that the process now given is simply an arrangement on paper of the process which before would pass through his mind without being taken note of.

Example 4. Divide 270 tons 17 cwt. 1 qr. 17 lb. 1 oz. by 135.

```
      tons  cwt. qr. lb. oz. tons cwt. qr. lb. oz.
135 ) 270   17   1   17  1 ( 2    0    0   14  7
      270
      ---
       17
        4
      ---
       69
       28
      ---
      569
      138
      ----
     1949
      135
      ----
      599
      540
      ---
       59
       16
      ---
      355
       59
      ---
      945
      945
      ---
```

Exercises. Set XXVI.

1. Divide £8003 4 $1\frac{3}{4}$ by 103.
2. Divide £689 4 $7\frac{1}{4}$ by 1001.
3. „ £1854 5 $7\frac{1}{2}$ by 347.
4. „ £2645 4 $4\frac{1}{2}$ by 141.
5. „ £3050 12 $10\frac{1}{2}$ by 154.
6. „ £7379 9 $0\frac{3}{4}$ by 465.
7. „ £171 19 $5\frac{1}{2}$ by 194.
8. „ £949 14 $6\frac{3}{4}$ by 79.
9. „ £2117 1 $2\frac{1}{2}$ by 29.
10. „ £1005 8 3 by 3972.
11. Divide 14 tons 19 cwt. 23 lb. by 47, and by 113.
12. „ 1 cwt. 1 qr. 9 lb. 5 oz. 4 dr. by 19.
13. „ 5 da. 30 min. 46 sec. by 98, and by 128.
14. „ 5 wk. 2 da. 8 hr. 1 min. by 37.
15. „ 4 fur. 35 po. 1 yd. by 57, and by 69.
16. „ 196 mi. 256 yd. by 192, and by 299.

17. Divide 542 ac. 3 ro. 37 sq. po. by 591, and by 398.
18. „ 262 sq. yd. 4 sq. ft. 71 sq. in. by 29.
19. „ 51 qr. 4 bus. 1 pk. by 194, and by 79.
20. „ 42 cub. yd. 18 cub. ft. 450 cub. in. by 67.

Owing to the ease with which the quotient is obtained in the division of an integer by a power of 10, the above process in the case of such divisors may be much simplified.

Example 5. Divide £231 17s. 6d. by 100, and £1034 7s. 6d. by 1000.

£	s.	d.
2,31	17	6
20		
6,37		
12		
4,50		
4		
2,00		

£	s.	d.
1,034	7	6
20		
,687		
12		
8,250		
4		
1,000		

Quotient, £2 6s. 4½d. Quotient, £1 0s. 8¼d.

If, however, the divisor, though large, be a product of small integers, the process followed in Examples 3 and 4 may be departed from. Thus, since $135 = 5 \times 3 \times 9$, the result in the case of Example 4 may be obtained as follows :—

tons	cwt.	qr.	lb.	oz.
5)270	17	1	17	1
3)54	3	1	25	13
9)18	1	0	17	15
2	0	0	14	7

Exercises. Set XXVII.

1. Divide £7770 2 1 by 100.
2. Divide £1396 15 5 by 100.
3. „ £5343 15 0 by 1000.
4. „ £13967 14 2 by 1000.
5. „ £688 10 10 by 1000.
6. „ £20010 8 4 by 10000.
7. Divide 95 tons 1 cwt. 3 qr. 4 lb. by 1000.
8. „ 24 qr. 4 bus. 3 pk. 1 gall. by 100.
9. „ £432 5 0 by 112 (*i.e.* $4 \times 4 \times 7$).

10. Divide £486 18 5¼ by 165.
11. „ 15 tons 11 cwt. 3 qr. 21 lb. by 49.
12. „ 1 cwt. 1 qr. 17 lb. 3 oz. by 20.
13. „ 5 da. 1 hr. 44 min. 33 sec. by 99.
14. „ 4 fur. 31 po. 4 yd. 6 in. by 56.
15. „ 271 sq. yd. 4 sq. ft. 138 sq. in. by 30.
16. „ 42 cub. yd. 1 cub. ft. 108 cub. in. by 66.

II. When reference is made to a concrete unit in both dividend and divisor.

Example 1. How often is £193 12s. 8½d. contained in £968 3s. 6½d.?

£968 3s. 6½d. ÷ £193 12s. 8½d.
= 464725 halfpence ÷ 92945 halfpence,
= 5.

Or,

= 929450 far. ÷ 185890 far.
= 5.

The two given quantities are expressed in terms of one common unit, which is so chosen that a single whole number is thus obtained as the expression for each quantity. The required division is then performed.

Example 2. How often is 3 qr. 12 lb. contained in 6 cwt.?

qr.	lb.	cwt.
3	12	6
28		4
36		24
6		28
96		192
		48

$$96)\overline{672}(7$$
$$\underline{672}$$

This is the actual process, but the following abridgment gives all that is necessary, and, what is important, is fully self-explanatory:—

6 cwt. ÷ 3 qr. 12 lb. = 672 lb. ÷ 96 lb.
= 7.

EXERCISES. SET XXVIII.

Perform the following divisions :—

1.	£0	11	10½ ÷ £0	3	11½.		2.	£187	11	10½ ÷ £0	2	10½.
3.	17 guineas		÷ £1	9	9.		4.	£102	12	6¾ ÷ £0	16	8¼.
5.	£1492	11	10½ ÷ £5	14	4½.		6.	£348	12	6¾ ÷ £8	10	0¾.
7.	£271	0	7½ ÷ £0	17	2½.		8.	£1216	0	7½ ÷ £11	11	7½.

9. 5 da. 30 min. 46 sec. ÷ 1 hr. 13 min. 47 sec.
10. 3 cwt. 3 qr. 27 lb. 15 oz. 12 dr. ÷ 7 lb. 13 oz. 12 dr.
11. 14 fur. 25 po. 3 yd. ÷ 3 po. 2 yd. 1 ft.
12. 542 sq. yd. 8 sq. ft. 142 sq. in. ÷ 9 sq. yd. 67 sq. in.
13. 42 cub. yd. 18 cub. ft. 450 cub. in. ÷ 17 cub. ft. 342 cub. in.
14. 103 qr. 2 pk. ÷ 2 bus. 1 gall.

63. PRODUCTS OBTAINED WITH THE HELP OF DIVISION.—After a knowledge of the process of division has been obtained, many multiplications can be performed more easily than before. Thus, if asked to multiply 6s. 8d. by 23840, we may reason as follows :—

£1 repeated 23840 times = £23840. Now 6s. 8d. is exactly the *third* part of £1 ; ∴* 6s. 8d. repeated 23840 times = the third part of £23840, *i.e.*, £23840 ÷ 3, or £7946 13s. 4d.

Here, instead of the long process of calculation used in multiplying 6s. 8d. by 23840, we have substituted a process which is very much shorter, and gives the same result, viz., dividing £23840 by 3.

Example 1. Multiply 2s. 6d. by 18041.

2s. 6d. × 18041 = £18041 ÷ 8
= £2255 2s. 6d.

The possibility of these simplifications lies in the fact that the multiplicands 6s. 8d. and 2s. 6d. are each contained an exact number of times in a pound ; or, to employ the usual term, are *aliquot parts* of a pound. In order, therefore, to take advantage of such short methods, the learner must be familiar with the aliquot parts of the various units

* For shortness' sake the symbol ∴ is used instead of the word *therefore*, which necessarily occurs often in reasoning.

of measurement. Thus, in connection with the pound, he should know that—

10s. is the *half.*	3s. 4d. the *sixth* part.
6s. 8d. the *third* part.	2s. 6d. the *eighth* part.
5s. the *fourth* part.	2s. the *tenth* part.
4s. the *fifth* part.	1s. 8d. the *twelfth* part.

EXERCISES. SET XXIX.

Find the following products:—

1. 10s. × 81643, 5s. × 4276.
2. 5s. × 13845, 4s. × 8402.
3. 4s. × 7681, 2s. × 68417.
4. 2s. × 19853, 2s. 6d. × 28.
5. 2s. 6d. × 7416, 3s. 4d. × 76.
6. 3s. 4d. × 16182, 6s. 8d. × 16.
7. 6s. 8d. × 31861, 1s. 8d. × 25.
8. 1s. 8d. × 10751, 1s. × 2167.

By repeated application of the principle employed in the preceding examples, we may deal with multiplications in which the multiplicand is not an aliquot part of a higher unit. Thus, to multiply 16s. 8d. by 2165, we may say—

10s. × 2165 = £2165 ÷ 2 = £1082 10 0
and 6s. 8d. × 2165 = £2165 ÷ 3 = £721 13 4

therefore, adding these two results, we have

16s. 8d. × 2165 = £1804 3 4

The amount of actual calculation here is small, viz., a division by 2, a division by 3, and an addition; and in practice, where it is not intended that what is set down shall be self-explanatory, very few figures would suffice. The *full* process may be arranged as follows:—

	£2165	= £1 × 2165.
10s. is ½ of £1 ∴	1082 10 0	= 10s. × 2165.
6s. 8d. is ⅓ of £1 ∴	721 13 4	= 6s. 8d. × 2165.
∴ by add^{n}.,	£1804 3 4	= 16s. 8d. × 2165.

Example 2. Multiply 12s. 9½d. by 18415.

		£18415			$= £1 \times 18415$.
10s. is $\frac{1}{2}$ of £1	∴	9207	10	0	$= 10s. \times 18415$.
2s. 6d. is $\frac{1}{4}$ of 10s.	∴	2301	17	6	$= 2s.\ 6d. \times 18415$.
3d. is $\frac{1}{10}$ of 2s. 6d.	∴	230	3	9	$= 3d. \times 18415$.
$\frac{1}{2}$d. is $\frac{1}{6}$ of 3d.	∴	38	7	$3\frac{1}{2}$	$= \frac{1}{2}d. \times 18415$.
∴ by add^n,		£11777	18	$6\frac{1}{2}$	$= 12s.\ 9\frac{1}{2}d. \times 18415$.

Here we view 12s. $9\frac{1}{2}$d. as being = 10s. + 2s. 6d. + 3d. + $\frac{1}{2}$d. There are, however, other ways in which it may be broken up to suit the process; such as 5s. + 4s. + 3s. 4d. + 5d. + $\frac{1}{2}$d., &c.

Exercises. Set XXX.

Find the following products:—

1. 12s. 6d. × 375.
2. 7s. 6d. × 191.
3. 13s. 4d. × 312.
4. 5s. 8d. × 176.
5. 8s. 4d. × 79.
6. 7s. 4d. × 999.
7. 17s. 6d. × 37.
8. 6s. 6d. × 797.
9. 18s. 4d. × 723.
10. 7s. 8d. × 92.
11. 3s. $6\frac{1}{2}$d × 27.
12. 5s. $10\frac{1}{2}$d. × 85.
13. 2s. $9\frac{1}{4}$d. × 87.
14. 12s. $9\frac{1}{4}$d. × 45.
15. 14s. $7\frac{3}{4}$d. × 117.
16. 6s. $9\frac{3}{4}$d. × 314.
17. 16s. $3\frac{1}{4}$d. × 29.
18. 19s. $4\frac{3}{4}$d. × 197.
19. 2s. $9\frac{3}{4}$d. × 417.
20. 7s. $11\frac{1}{4}$d. × 315.

If in the multiplicand the highest unit itself occurs, we must proceed as in the following examples:—

Example 1. Find the cost of 1107 articles at £10 9s. $7\frac{1}{2}$d.

		£1107			= cost at £1 each.
Mult^g by		10			
we have		11070	0	0	= cost at £10 each.
5s. is $\frac{1}{4}$ of £1	∴	276	15	0	= „ 5s. „
4s. is $\frac{1}{5}$ of £1	∴	221	8	0	= „ 4s. „
6d. is $\frac{1}{8}$ of 4s.	∴	27	13	6	= „ 6d. „
$1\frac{1}{2}$d. is $\frac{1}{4}$ of 6d.	∴	6	18	$4\frac{1}{2}$	= „ $1\frac{1}{2}$d. „
∴ by add^n,		£11602	14	$10\frac{1}{2}$	= cost at £10 9s. $7\frac{1}{2}$d. each.

Example 2. What is the total weight of 315 coal waggons which weigh each 6 tons 16 cwt. 3 qr.?

	tons	cwt.	qr.	
	315	0	0	= weight at 1 ton each.
Mult^g by			6	
we have	1890	0	0	= weight at 6 tons each.
10 cwt. is $\frac{1}{2}$ of 1 ton ∴	157	10	0	= " 10 cwt. "
5 cwt. is $\frac{1}{2}$ of 10 cwt. ∴	78	15	0	= " 5 cwt. "
1 cwt. is $\frac{1}{5}$ of 5 cwt. ∴	15	15	0	= " 1 cwt. "
2 qr. is $\frac{1}{2}$ of 1 cwt. ∴	7	17	2	= " 2 qr. "
1 qr. is $\frac{1}{2}$ of 2 qr. ∴	3	18	3	= " 1 qr. "
∴ by add^n,	2153	16	1	= weight at 6 tons 16 cwt. 3 qr. each.

It will now be quite evident that this method may be followed in *every* case where the multiplicand is expressed in terms of several units, and the multiplier is an integer; but it must also be clear that the more divisions we have to perform in any instance, the less advantage has the method over the ordinary processes of multiplication.

EXERCISES. SET XXXI.

1. £3 12 6 × 114.
2. £7 7 6 × 127.
3. £2 13 4 × 152.
4. £3 5 8 × 342.
5. £5 8 4 × 371.
6. £5 17 6 × 175.
7. £1 18 6 × 134.
8. £2 13 8 × 317.
9. £7 12 10 × 782.
10. £8 13 $7\frac{1}{2}$ × 374.
11. £3 15 $10\frac{1}{2}$ × 115.
12. £6 13 $6\frac{1}{4}$ × 217.
13. £2 14 $9\frac{3}{4}$ × 371.
14. £7 8 $11\frac{1}{2}$ × 397.
15. £3 17 $6\frac{1}{4}$ × 289.
16. £7 11 $8\frac{3}{4}$ × 184.
17. £15 18 $9\frac{1}{4}$ × 154.
18. £10 17 $11\frac{3}{4}$ × 235.
19. £11 15 $11\frac{3}{4}$ × 276.
20. £23 18 $11\frac{3}{4}$ × 713.

Sometimes it is of advantage to view the multiplicand not as the sum of several items, but as a difference in which the subtrahend is an aliquot part of the highest unit. Thus looking upon £4 19s. as £5 − 1s., we may find the result of multiplying £4 19s. by 3147 in the following way:—

£5 × 3147 =	£15735	0	0
1s. × 3147 = £3147 ÷ 20 =	157	7	0

And subtracting we have

£4 19s. × 3147 = £15577 13 0

Exercises. Set XXXII.

Find the following products :—

1. 17s. 6d. × 394.
2. 16s. 8d. × 217.
3. 19s. 6d. × 377.
4. 18s. 4d. × 399.
5. 9s. 2d. × 372.
6. 6s. 7d. × 481.
7. £9 19 4 × 117.
8. £19 17 6 × 315.
9. £99 19 10½ × 714.
10. £49 18 11 × 375.

Examination Papers on §§ 48—63.

I.

1. What are the standard units of time, length, and mass? Explain how each of them is defined or fixed.

2. Express £71 0s. 10½d. in *halfpence*, and 39103 farthings in the usual way of expressing a sum of money.

3. From the sum of 6 ac. 37 sq. po. and 13 ac. 3 ro. 19 sq. po., subtract 9 ac. 2 ro. 29 sq. po.

4. Multiply 653 cub. yd. 11 cub. ft. 1332 cub. in. by 11, and divide the result by 12.

5. How often is 5 lb. 7 oz. contained in 6 cwt. 2 qr. 22 lb. 6 oz.?

6. Which is the greater, and by how much, 857 fathoms or 1 mi. 3 fur.?

7. Starting with £10, I pay a railway fare of 6s. 3d., and purchase 50 lb. of tea at 2s. 11d. per lb., and 11 yd. of ribbon at 1s. 7½d. per yard. What sum must I then have remaining?

II.

1. Show by a diagram that there being 12 inches in a foot there must be 144 square inches in a square foot. Explain what is meant by a cubic yard.

2. Express 19 ac. 32 sq. po. in *square yards*, and 3170802 oz. (avoirdupois) in *tons*, &c.

3. In a barrel of beer there are 36 gallons. Find the number of half-pints in 3 barrels.

4. If 13 times the ninety-ninth part of 41 mi. 6 fur. 5 po. be taken, what is the result?

5. Explain and illustrate by an example the process of finding the number of poles in a number of yards.

6. If I have £10 1s. 4½d., to how many persons can I give the sum of 3s. 5½d.?

7. A clock gains exactly 1 min. 47 sec. every day. How much will it gain in 6 weeks?

III.

1. Explain and illustrate by an example the disadvantages attending the use of our present units of measurement.

2. Express 17294544 sq. in. in *acres*, &c., and 1 cwt. 3 lb. (avoirdupois) in *troy pounds*, &c.

3. How often must a wheel 11 ft. 5 in. in circumference turn round in going 2 mi. 1275 yd.?

4. Find by three different methods the cost of 573 acres of land at £3 17s. 4d. per acre.

5. Multiply by 4 the excess of a mean solar year over 365 days, and subtract the result from 1 day.

6. One cask of whisky contains 42 gall. 2 qt. 1 pt., a second 40 gall. 3 qt., and a third 29 gall. The contents of the three are divided among 29 persons. How much does each receive?

7. If I present £20 in payment for 7 cwt. 1 qr. 17 lb. of sugar at $5\frac{1}{2}$d. per lb., what change should be returned to me?

PRACTICAL EXAMPLES INVOLVING BOTH MULTIPLICATION AND DIVISION.

64. Were we to put the question—*If 7 cwt. of sugar cost £13 1s. 4d., what is the price per cwt.?* the learner would have now no difficulty in finding the answer. He would proceed thus:—

Price of 7 cwt. = £13 1s. 4d.
therefore, price of 1 cwt. = seventh part of £13 1s. 4d.
= £13 1s. 4d. ÷ 7
= £1 17s. 4d.

Again, were we to ask—*If the price of sugar per cwt. be £1 17s. 4d., what would 9 cwt. cost?* he would say:—

Price of 1 cwt. = £1 17s. 4d.
therefore, price of 9 cwt. = 9 times £1 17s. 4d.
= £1 17s. 4d. × 9
= £16 16s.

From these examples, therefore, it may be seen that he is able to answer a more difficult question than either, viz.:—

If 7 cwt. of sugar cost £13 1s. 4d., what would 9 cwt. cost?
For, knowing the cost of 7 cwt., he is able to find, as above, the cost of 1 cwt.; and then, knowing the cost of 1 cwt., he can find the cost of 9 cwt.

Example 1. A railway train going at a constant speed travels 285 mi. 1350 yd. in 15 hours. How far would it go in 7 hours?

In 15 hours it travels 285 mi. 1350 yd.
∴ in 1 hour " 285 mi. 1350 yd. ÷ 15,
i.e. 19 mi. 90 yd.
∴ in 7 hours " 19 mi. 90 yd. × 7
i.e. 133 mi. 630 yd.

Example 2. A pedestrian travelling 4 hours each day performs a certain journey in 21 weeks. How many weeks would he take travelling at the same rate during 7 hours per day?

Travelling 4 hr. per day he requires 21 weeks
∴ " 1 " " " " 21 wk. × 4
or 84 weeks
and ∴ " 7 " " " " 84 wk. ÷ 7
or 12 weeks.

In these and similar exercises the result evidently is reached in *two* steps. When the question was, "What would 9 cwt. cost?" we made the first step by finding what 1 cwt. would cost; when asked "How far would it go in 7 hours?" our first step was to find how far it would go in 1 hour; and such is the mode of procedure in all similar cases. In other words, we first calculate the *rate* (at which the sugar is sold, or at which the train goes), and thence proceed to calculate the result wanted.

Exercises. Set XXXIII.

1. 15 yd. of cloth cost £1 3s. 1½d. What would 37 yd. cost at the same rate?

2. The rent of a field containing 7 ac. is £8 3s. 4d. At this rate what would the rent of 100 ac. be?

3. A pumping-engine raises 17 gall. 2 qt. of water in 10 seconds. Working at the same rate, how much would it raise in 66 seconds?

4. If 12 tons 13 cwt. of coals cost £11, how much may be bought for £17?

5. What must be paid for 33 doz. of wine at the rate of £105 for 120 doz.?

6. How many acres of land may be rented for £153 at the rate of £18 for 6 ac. 3 ro. 36 po.?

7. If 5 men mow a certain quantity of hay in 14 hours, how long would 12 men take to do the same, all the men working at the same rate?

8. The tax upon a house-rent of £175 is £7 5s. 10d. At this rate what tax must one pay in the case of a house rented at £33?

9. What must sheep cost per score at the rate of £176 19s. 3d. for 121?

10. A pedestrian walking at the rate of 117 yd. per minute performs a certain journey in 6 hours. How long would he take if he travelled at the rate of 52 yd. per minute?

65. The learner is already familiar with the fact that when one number is to be divided by another and the quotient to be multiplied by a third, the same final result may be got, and in many cases more readily, by performing the operations in the reverse order. This should be steadily borne in mind in dealing with exercises like the present.

Example. A field of 26 ac. is rented for £62. What would be the rent of 200 ac. at the same rate?

$$
\begin{array}{lllrl}
 & \text{Rent of} & 26 \text{ ac.} & = & \pounds 62 \\
\therefore & ,, & 1 \text{ ac.} & = & \pounds 62 \div 26 \\
\text{and } \therefore & ,, & 200 \text{ ac.} & = & (\pounds 62 \div 26) \times 200 \\
 & & & = & (\pounds 62 \times 200) \div 26 \\
 & & & = & \pounds 12400 \div 26 \\
 & & & = & \pounds 476\ 18\text{s. } 5\tfrac{1}{2}\text{d. } \tfrac{2}{13}.
\end{array}
$$

Here, instead of dividing by 26 and multiplying the result by 200, we multiply by 200 and divide the result by 26—an easier way of reaching the same conclusion.

EXERCISES. SET XXXIV.

1. If 15 cwt. of sugar cost £28, what will 31 cwt. cost at the same rate?

2. If 16 tons of iron cost £42, what will 157 tons cost at the same rate?

3. For a railway journey of 288 mi. the fare is £3. At this rate what would the fare be for a journey of 517 mi.?

4. What would 119 yd. of cambric cost at the rate of £42 for 36 yd.?

5. 6960 sq. yd. of matting cost £270. How much of it may be bought for £1000 at this rate?

6. A person who has used 48 thousand cub. ft. of gas is charged £11 for it. What must another consumer pay who has used 37 thousand cub. ft.?

7. How much wine may be bought for £173 at the rate of £40 for 50 gallons?

8. How many acres of land may be rented for £763 at the rate of 105 ac. for £112?

9. A machine makes 24 cwt. of nails in 37 hours. Working at this rate, how long will it take to produce 115 cwt.?

10. Two tradesmen have respectively £480 and £875 invested in the same business, and the investor of the former sum receives £31 as his share of the profits. What must the other receive?

66. In all these questions it must have been observed that there are given two magnitudes of the same kind, to one of which there corresponds in some way a third given magnitude, and to the other in the same way a fourth magnitude *not given*, but to be found. For example, in No. 4 of the questions immediately preceding we have two given magnitudes of the same kind, viz., "36 yd." and "119 yd.," to the former of which there is given a corresponding magnitude, viz., its price, and the magnitude corresponding in the same way to the latter the learner is asked to find.

In the preceding examples the two given magnitudes of the same kind have always appeared in terms of one and the same unit. Thus in (9) it is the *hundredweight*, in (3) the *mile*, and so on. The point now requiring the learner's attention is that if they be not thus expressed when given, the necessary change must at once be made.

Example. If 27 cwt. of coal cost £1 9s. 3d., how much may be bought for 4s. 4d.?

Here the two given magnitudes of the same kind, viz., £1 9s. 3d. and 4s. 4d., are not in terms of one and the same

unit, and we therefore proceed to express them both in pence alone. Thus—

$$\pounds 1\ 9s.\ 3d. = 351 \text{ pence}$$
$$4s.\ 4d. = 52 \text{ pence.}$$

$$\begin{array}{llll}
\text{Now} & 351d. \text{ is the price of} & & 27 \text{ cwt.} \\
\therefore & 1d. \quad ,, & ,, & 27 \text{ cwt.} \div 351 \\
\text{and} \therefore & 52d. \quad ,, & ,, & (27 \text{ cwt.} \div 351) \times 52 \\
& & \text{which} & = (27 \text{ cwt.} \times 52) \div 351 \\
& & & = 1404 \text{ cwt.} \div 351 \\
& & & = 4 \text{ cwt.}
\end{array}$$

EXERCISES. SET XXXV.

1. If the cost of 3 bus. 3 pk. of wheat be £1 1s. 3d., what will be the cost of 5 qr. 7 bus. at the same rate?

2. The price of 1 ton 13 cwt. of coal is £1 18s. 6d. What will 3 tons 12 cwt. cost at the same rate?

3. Going at a constant speed a railway train accomplishes 97 mi. in 3 hr. 14 min. At this rate how far would it go in 7 hr. 36 min.?

4. If 1 ton 13 cwt. of coal cost £1 18s. 6d., how much may be bought for £5 19s.?

5. Going at the rate mentioned in (3) how long would a train take to run a distance of 117 mi. 44 yd.?

6. How much land may be rented for £50 8s. 4d. at the rate of £71 11s. 10d. for 3 ac. 2 ro. 8 sq. po.?

7. The weight of 21 pints of water is 26 lb. 4 oz. What will 7 gall. 3 qt. of the same water weigh?

8. At the rate mentioned in (6) what would the rent of 17 ac. 310 sq. yd. be?

9. A vertical rod 2 ft. 3 in. long throws a shadow 31 ft. 6 in. long. What must the length of a flagstaff be which at the same time throws a shadow 98 yd. long?

10. A bankrupt pays £2 6s. 8d. to a creditor to whom he owed £40. What must he pay to another to whom he owed £113 2s. 6d.?

11. One gentleman's income is £1152 17s., and another's £3174 14s. 6d. If the former pays £96 1s. 5d. of income-tax, what must the latter pay?

12. The occupier of a house whose rental is £81 4s. is charged 15s. 2½d. for the school board; another is charged £1 0s. 4½d. What rent must the latter pay?

67. Many questions, as easy as any of the foregoing, may present difficulty to the learner on account of being stated in a less direct way, or in technical language with which he is unacquainted. All the guidance that can be given in

reference to them is the general advice to make sure of the meaning of every word and phrase used, and, separating clearly the information given from what is desired to be known, to restate the question in the simplest form possible.

EXERCISES. SET XXXVI.

1. What is the dividend on £6180 10s. at 5 per cent.?

2. A bankrupt whose debts amount to £3177 pays £3 18s. 4d. to a creditor whose account is £35. What is the total value of the bankrupt's effects?

3. A bankrupt's debts amount to £8405, and his effects are worth only £3221 18s. 4d. What loss will a creditor sustain whose account is £105 2s. 6d.?

4. After income-tax at the rate of 5d. in the pound has been deducted from a gentleman's income there remains £979 3s. 4d. What must his income be?

5. After income-tax at the rate of 7d. in the pound has been deducted from a gentleman's income there remains £1902 16s. 8d. What is the amount of his tax?

6. A man travels 559 miles in 43 days walking 4 hours a day. At this rate how many miles would he travel in 22 days walking 6 hours a day?

7. 108 men are started upon a piece of work which they can finish in 24 days; when half of it is done 54 additional men are engaged. In how many days after this will the work be completed?

EXERCISES. SET XXXVII.

Miscellaneous.

1. The difference of two numbers is nine hundred and nine, and the larger is two thousand and forty. Find the other.

2. How often is the fourth power of 8 contained in the eighth power of 4?

3. One parcel weighs 2 lb. 5 oz. and another five times this. What is the weight of both together?

4. £13 17s. 6d. was expended in paying 1s. 3d. each to a number of workmen. Find how many workmen there were?

5. One piece of cloth contains 27 yd. at 1s. $7\frac{1}{2}$d. per yd., and another 35 yd. at 1s. $3\frac{3}{4}$d. per yd. What is the difference in the price of the pieces?

6. A contractor employs a gang of 327 navvies, each of whom receives 3s. 2d. a day. In 15 days what will the wages of the gang amount to?

7. A person who owed £315 16s. 4d. agreed to pay it in monthly instalments of £4 19s. 6d. After 57 payments how much does he owe?

8. After paying for 13 yd. of cloth at 12s. $6\frac{1}{4}$d. a yd. I have £2 10s. 4d. remaining. What sum had I before?

9. A farmer sells 26 oxen valued at £11 12s. 6d. each, and then buys 150 sheep at 27s. 6d. each. What balance of money remains in hand?

10. In a factory 215 persons are employed, 195 of whom receive 2s. 4d. a day each, and the others 6d. more. What is the total sum paid to them daily?

11. Every sheep in 15 score cost £1 7s. 6d. What was the total cost?

12. If the cost of 117 doz. of handkerchiefs be £32 3s. 6d., what is the cost of one handkerchief?

13. A person possessing 130 ac. of land sells a corner of it containing 10000 sq. yd. How many acres has he remaining?

14. A person whose fortune amounted to £6745 10s. left one half of it to be divided equally among six nephews, and the other half equally among four nieces. What would each nephew and niece receive?

15. In a book of 315 pages there are 42 lines on each page. If the author be paid at the rate of $1\frac{1}{4}$d. a line, what sum should he receive?

16. What will 3 cwt. 1 qr. 17 lb. of sugar cost at $5\frac{1}{2}$d. per lb.?

17. At the rate of 15s. $4\frac{1}{2}$d. for a napoleon, what sum should I receive on changing 17 napoleons and 3 half-napoleons into English money?

18. A draper buys 13844 yd. of cloth for £660 19s. 4d. and sells it for £733 1s. 5d. What does he gain per yd.?

19. The sum of two numbers is two thousand and sixteen, and the one exceeds the other by ninety. Find the numbers.

20. In payment of a sum of 10 guineas I gave 15 half-sovereigns, 15 half-crowns, and the remainder in sixpences. Find the number of sixpences.

21. 17 casks of treacle weigh 56 cwt. 1 lb., and the nett weight after deducting the weight of the casks is 54 cwt. 4 lb. What is the average weight of an empty cask?

22. The double of a fifth part of a number is thirty thousand and ten. Find the number.

23. A farmer who bought 17 oxen at £16 13s. 6d., and 210 sheep at 25s. 3d. each, sold each ox for £18 and each sheep for 23s. 10d. What did he thereby gain or lose?

24. A person is found to take 119 steps in a minute. At this rate how far will he go in 3 hours, supposing the length of each step to be 2 ft. 8 in.?

25. The sum of a certain number and forty score is a thousand dozen. Find the number.

26. For 1128 bottles of wine I pay £209 3s. What is the rate per doz.?

27. A wheel going constantly for 7 weeks turns 22226400 times. How many times does it turn in a minute?

28. The sum of three hundred and one and nine times a certain number is a million and five. Find the number.

29. In a granary there are stored 71235 bushels of corn. What is the value of it at the rate of 17s. 6¼d. per sack of 5 bushels?

30. Seven times a certain number is greater than ten thousand four hundred by eight hundred and seven. Find the number.

31. A tract of land containing 500 ac. 2 ro. is divided among 89 persons, of whom 49 receive each 3 ac. 1 ro. 7 sq. po., and the others receive the remainder in equal lots. What is the size of one of these lots?

32. In dividing a certain number by 39 the quotient is 207, with 27 remaining still to be divided. Find the dividend.

33. In dividing 9009 by a certain number the quotient is 310, with 19 remaining to be divided. Find the divisor.

34. One cask contains 75 gall. 3 qt. 1 pt. of wine and another 64 gall. 2 qt. 1 pt. How much must be transferred from the one to the other so as to have the same quantity in each?

35. The fourth part of a certain number is multiplied by seventeen, the result being thirteen hundred and nine. Find the number.

36. One man's wage is 4s. 2d. a day, and another's 33s. 6d. a week. If the former work six days a week, what will be the difference in their incomes in 47 weeks?

37. A ninth part of the sum of nineteen hundred and nineteen and a certain number is nine thousand and ninety-one. Find the number.

38. A pipe conveys 1 gall. 3 qt. of water every second into a basin which is capable of holding 8916 gall. 3 qt. What time will be required to fill the basin in this way?

39. A man whose quarterly expenditure amounts on an average to £49 7s. 6d. saves 750 guineas in 15 years. What is his annual income?

40. A certain number is multiplied by seventy-one and the product divided by seventeen, the quotient obtained being a million and thirty-five. Find the number.

41. The land possessed by two proprietors amounts to 512 ac. 3 ro. 24 sq. po. After the one has given the other 16 ac. 35 sq. po. their portions are equal. How many acres did each previously possess?

42. The engines of a factory consume 364 cwt. of coal per day, and in a year of 300 working days the expense in this way for coal was £6370. What did it cost per ton?

43. A certain number divided by 508 gives the same quotient as 2604315 divided by 357. Find the number.

44. A bankrupt's effects are worth £35700 and his debts amount to £61200. What will he pay in the pound?

45. One pipe supplies 1046078 cub. in. of water in 17 hours, and another 935845 cub. in. in 19 hours. How much more per hour does the one pipe supply than the other?

46. A person sells 573 yd. of cloth at 5s. 7½d. per yd., and thus gains £3 11s. 7½d. What did it cost him per yd.?

47. A boy divides a certain number by 9, increases by 9 the quotient thus obtained, multiplies this sum by 9, and subtracts 9 from the product, his final result being 198. What number did he start with?

48. For 7 barrels of beer each containing 36 gallons I pay 12 guineas. What does it cost per pint?

49. A woman buys 19 doz. of eggs at 10½d. per doz., loses 36 eggs by breakage, and sells the remainder at 1s. 0¾d. per doz. What does she gain or lose by the transaction?

50. A waggon full of coal weighs 8 tons 1 qr., and the tare (*i.e.*, in this case, the weight of the empty waggon) is 2 tons 15 cwt. 2 qr. Find the cost of the coal at £1 1s. 8d. per ton.

51. 3 cwt. 3 qr. 13 lb. of raisins are to be packed in 17 boxes, so that one box may contain 9 lb. less than each of the others. How much is put into each of the 16 boxes?

52. 17 bushels of corn are worth £2 19s. 6d., and 100 bushels of wheat are worth £30 12s. 6d. How many bushels of wheat must be given in exchange for 133 bushels of corn?

53. The sum of two numbers is 3948, and the quotient obtained on dividing the greater by the less is 19, with 108 still to be divided. Find the numbers.

54. A certain number of threepenny pieces, the same number of fourpenny pieces, and the same number of crowns, are in all equivalent to £67. How many coins are there altogether?

55. One traveller whose rate of walking is 4 mi. 1430 yd. per hour leaves a town 3 hours after another whose rate is 3½ miles per hour. In what time will the second overtake the first?

56. A person buys eggs at 10s. 5d. a hundred, and sells them at 1s. 4½d. a doz., thereby gaining 6s. 3d. How many eggs did he buy?

57. While the small wheel of a carriage turns 17 times, the large one, whose circumference is 12 ft. 6 in., turns 11 times. How many turns will the small wheel make in going a distance of 77 mi. 513 yd. 1 ft.?

58. An agent buys 213 ac. of land at £4 6s. 6d. per acre, and resells 194 ac. at £4 5s. 8d. per acre. At how much per acre must he sell the remainder so as to realise a clear gain of £96 17s. 10d.?

59. A wine merchant buys 314 gall. of wine at 16s. 6d. per gallon, and 273 gall. at £1 0s. 4d. per gallon, and mixes the two quantities, losing in the process 7 gallons. What will he gain by selling the mixture at 19s. 10d. per gallon?

60. In a constituency 10405 electors voted for two candidates, and the successful candidate had a majority of 289, his election expenses being £10705 2s. 9½d., while his opponent's were only £5690 5s. What on an average did a single vote cost each candidate?

INTEGRAL NUMBERS VIEWED AS PRODUCTS OF OTHER INTEGERS.

68. An acquaintance with the mutual relations of numbers, especially integers, and with their properties is of much importance to the calculator in enabling him to shorten and simplify his work. This truth we have already seen illustrated in some very simple instances. For example, he who knows that 125 is the eighth part of 1000 can attain the result of multiplying or dividing by 125 with greater ease than one who is ignorant of the existence of this relation between the two numbers. For the reason stated we propose, in what immediately follows, to increase the learner's knowledge of properties of integral numbers and relations between them of the kind indicated in this example.

69. Every multiplication we perform teaches us a property of the number which results from the operation. For example, we obtain 483 on multiplying 7 by 69, and we thus know, what probably we were before ignorant of, that 483 is exactly divisible by two integers other than itself and unity, viz., by 7 and by 69. That *very many* integers have this property is a fact we can therefore assert from experience. But on trial of the small integers in order, we find at once that it is not true of *all*: for example, 7 is exactly divisible by no integer except itself and 1. We may consequently divide integral numbers into two classes, those which have this property and those which have not; the former are called *composite* numbers, *e.g.*, 4, 6, 8, 9, . . . , and the latter *prime* numbers, *e.g.*, 2, 3, 5, 7, . . .

70. Many facts regarding this subject of exact divisibility are soon discovered by the learner himself, *e.g.*:—

(1.) A number is known to be *even*, *i.e.*, exactly divisible by 2, if it have 0, 2, 4, 6, or 8 for its last digit.

(2.) A number is known to be exactly divisible by 5 if it have either 0 or 5 for its last digit.

(3.) A number is known to be exactly divisible by 10 if it have 0 for its last digit; by 100 if it have its last two digits each 0; by 1000 if it have its last three digits each 0; and so on.

Less likely to strike one are the following properties :—

(4.) A number is exactly divisible by 4 if the part of it represented by its last two digits be so divisible.

(5.) A number is exactly divisible by 8 if the part of it represented by its last three digits be so divisible.

Still less evident properties are the following :—

(6.) A number is exactly divisible by 3 or by 9 if the sum of its digits be so divisible.

(7.) A number is exactly divisible by 11 if the difference of the sum of the digits in the odd places and the sum of the digits in the even places be 0 or exactly divisible by 11.

71. When we find two or more integers other than unity whose product equals a given composite number we are said to have *resolved* the given number *into factors*. Thus, knowing that 395 is exactly divisible by 5 we may perform the division, and so resolve 395 into two factors 5 and 79. Having resolved a composite number into factors we may find that these factors are all prime numbers, and if so, we say that the given number has been *resolved into prime factors*, *e.g.*, $15=3\times5$. But as any factor which is not prime may be itself resolved into factors, and so on until resolution is no longer possible among them, we see that *every* composite number can be resolved into prime factors, *e.g.*—

$$\begin{aligned}120&=12\times10\\&=4\times3\times2\times5\\&=2\times2\times3\times2\times5.\end{aligned}$$

Further, it can be shown that for any given number only one set of such factors is possible.

72. With the knowledge of the properties given in § 70 it is easy to resolve many composite numbers into their prime factors.

Example. Resolve 403920 into its prime factors.

$$\begin{aligned}403920 &= 10 \times 40392 \\ &= 2 \times 5 \times 8 \times 5049 \\ &= 2 \times 5 \times 2 \times 2 \times 2 \times 11 \times 459 \\ &= 2 \times 5 \times 2 \times 2 \times 2 \times 11 \times 9 \times 51 \\ &= 2 \times 5 \times 2 \times 2 \times 2 \times 11 \times 3 \times 3 \times 3 \times 17.\end{aligned}$$

Here, observing that the given number ends in 0, we resolve it in the first line into 10 and another factor. In the second line 10 is resolved into 2 and 5, and observing that 392 is divisible by 8 we resolve 40392 into 8 and another factor. In the third line we first resolve 8 into its prime factors, then observing that the sum of the digits in the even places of 5049 is 9, and in the odd places also 9, and therefore the difference of the two sums 0, we resolve 5049 into 11 and another factor; and so on.

It is easily seen, however, that resolution into prime factors may sometimes be a matter requiring very great time and labour. If, for instance, the given number were 9211, we soon know from § 70 that it is not exactly divisible by 2, 3, 5, or 11; but as for other prime numbers, 7, 13, 17, &c., there is no course open to us except making trial with each of them by the ordinary process of division. In this laborious way we find that the first prime by which the given number is exactly divisible is 61, the quotient being 151. After a little more trouble we find that 151 is a prime, and thus conclude that the prime factors of 9211 are 61 and 151.

Exercises. Set XXXVIII.

Resolve the following numbers into prime factors:—

1. 100, 32, 72.
2. 144, 63, 180.
3. 75, 125, 175.
4. 117, 153, 207.
5. 57, 87, 147.
6. 255, 441, 555.
7. 765, 1128, 516.
8. 374, 517, 1045.

9. 2376, 8415. 10. 77517, 275517.
11. 377, 221, 323. 12. 2527, 1183, 6877.
13. Find all the prime numbers between 50 and 100.
14. Find all the prime numbers between 160 and 180.

73. When we know the prime factors of a number we are able to tell every one of the integers by which it is exactly divisible. Thus, knowing that the prime factors of 42 are 2, 3, 7, we thence know that 42 is exactly divisible also by 2×3, *i.e.* 6, by 2×7, *i.e.* 14, and by 3×7, *i.e.* 21, and by no other integers except 42 itself and unity. These numbers,

$$1, 2, 3, 6, 7, 14, 21, 42,$$

are called *measures* of 42, a measure of a number being defined as a number which is contained in it an exact number of times.

As another instance, we may find all the integral measures of 315. The prime factors of 315 are easily seen to be 3, 3, 5, 7, and thence we have

3×3 *i.e.* 9	$3 \times 3 \times 5$ *i.e.* 45
3×5 *i.e.* 15	$3 \times 3 \times 7$ *i.e.* 63
3×7 *i.e.* 21	$3 \times 5 \times 7$ *i.e.* 105
5×7 *i.e.* 35	

the full list of integral measures thus being

$$1, 3, 5, 7, 9, 15, 21, 35, 45, 63, 105, 315.$$

74. If we compare these two lists, the integral measures of 42 and those of 315, we see that the numbers

$$1, 3, 7, 21$$

are common to both, and we therefore say that 1, 3, 7, 21 are *common measures* of 42 and 315; a common measure of two or more numbers thus being a number which is contained in each of the numbers an exact number of times. Further, of these four integral measures, 1, 3, 7, 21, common to 42 and 315, the greatest is 21, which is in consequence naturally called the *greatest common measure* of 42

and 315, or, for shortness' sake, the G. C. M. of 42 and 315.

Similarly, finding the integral measures

of 12 to be 1, 2, 3, 4, 6, 12

of 18 to be 1, 2, 3, 6, 9, 18

and of 30 to be 1, 2, 3, 5, 6, 10, 15, 30,

we say that the common measures of 12, 18, and 30 are

1, 2, 3, 6,

and that the greatest common measure is 6.

75. Should one of two given numbers be a measure of the other, the former is of course the greatest common measure of the two. Thus, 12 being a common measure of 12 and 36, and being the greatest possible measure of 12, must therefore be the greatest common measure of 12 and 36.

76. It is often, as will be seen, of considerable importance to be able to find readily the greatest common measure of two numbers. For this purpose the method used above may be shortened as in the following example:—

Example. Find the greatest common measure of 84, 540, and 330.

$$84 = 2 \times 2 \times 3 \times 7,$$
$$540 = 2 \times 2 \times 3 \times 3 \times 3 \times 5,$$
$$330 = 2 \times 3 \times 5 \times 11,$$
$$\therefore \text{ G. C. M.} = 2 \times 3$$
$$= 6$$

Here, after resolving each of the numbers into its prime factors, we do not as before find all the integral measures of each of the numbers, and then select the largest which is common, but, as a short way to this, we at once take the greatest product of prime factors which is seen to be contained in all the numbers. That is to say, we look at the numbers in their resolved forms and observe that 2×3 occurs in all, and that this cannot be said of any product of a greater number of prime factors, and we thus conclude that 2×3 is the greatest common measure sought for.

As other instances we have from the above—

G. C. M. of 84 and $540 = 2 \times 2 \times 3 = 12$

G. C. M. of 540 and $330 = 2 \times 3 \times 5 = 30$.

EXERCISES. SET XXXIX.

Find the greatest common measure of—

1. 18, 42.	2. 42, 49.
3. 75, 225.	4. 12, 18, 45.
5. 30, 39, 72.	6. 64, 84, 144.
7. 504, 720.	8. 14, 18, 36, 48.
9. 459, 594.	10. 693, 1287.

77. Resolution into prime factors being often a work of great labour, the method followed in the preceding exercises is not generally serviceable. It can, however, be shown that *if the greater of two integers be divided by the other, the greatest common measure of divisor and dividend is the same as the greatest common measure of remainder and divisor;* and from this principle a better method is easily arrived at. Thus, if the greatest common measure of 299 and 221 be wanted, we divide 299 by 221,

```
221)299(1
    221
    ---
     78
```

and hence know that the greatest common measure of the divisor and dividend, 221 and 299, is the same as the greatest common measure of the remainder and divisor, 78 and 221. This latter may therefore be sought for instead.

Well, dividing 221 by 78,

```
78)221(2
   156
   ---
    65
```

we learn in the same way that the greatest common measure of 78 and 221 is the same as the greatest common measure of 65 and 78; and this which is more easily obtained we now seek for.

Dividing then again,

$$\begin{array}{l} 65)78(1 \\ \underline{65} \\ 13 \end{array}$$

we know that the greatest common measure of 65 and 78 is the same as the greatest common measure of 13 and 65, which, as the numbers are smaller, is still more easily found.

Dividing once more,

$$\begin{array}{l} 13)65(5 \\ \underline{65} \end{array}$$

we see that the greatest common measure of 13 and 65 is 13, and 13 is thus the greatest common measure also of the given numbers 221, 299.

The process here followed may be arranged thus :—

$$\begin{array}{l}
\phantom{\text{G. C. M.}=}221)299(1 \\
\phantom{\text{G. C. M.}=221)}\underline{221} \\
\phantom{\text{G. C. M.}=221)2}78)221(2 \\
\phantom{\text{G. C. M.}=221)278)}\underline{156} \\
\phantom{\text{G. C. M.}=221)278)1}65)78(1 \\
\phantom{\text{G. C. M.}=221)278)165)}\underline{65} \\
\phantom{\text{G. C. M.}=221)278)1}\text{G. C. M.}=13)65(5 \\
\phantom{\text{G. C. M.}=221)278)1\text{G. C. M.}=13)}\underline{65}
\end{array}$$

the last divisor being always the greatest common measure of the given numbers.

78. We may fitly describe this method of finding the answer to the question regarding the greatest common measure of two numbers as one of continued substitution of a simpler question. This will be more easily seen from the following explanatory abbreviation of the process which is given above in detail.

$$\begin{aligned}
\text{G. C. M. of } 221 \text{ and } 299 &= \text{G. C. M. of } 78 \text{ and } 221 \\
&= \text{G. C. M. of } 65 \text{ and } 78 \\
&= \text{G. C. M. of } 13 \text{ and } 65 \\
&= 13.
\end{aligned}$$

Example 2. Find the greatest common measure (1) of 3528 and 40488, (2) of 87 and 101.

```
3528)40488(11                      87)101(1
     3528                             87
     5208                             14)87(6
     3528                                84
     1680)3528(2                          3)14(4
          3360                              12
G. C. M.=168)1680(10                         2)3(1
             1680                              2
                                   G. C. M.=1)2(2
                                              2
```

79. Two integers, such as 87 and 101, which have no common measure except unity, are said to be *prime* to each other. The learner must consequently distinguish between two uses of the word "prime" in arithmetic; first, as applied to a single number, and, secondly, as applied to two numbers with the modifying phrase "to each other," or some equivalent word added. Thus 11 is a *prime number* or *prime;* 4 and 15 are *prime to each other*, *mutually prime* or *relatively prime.*

Exercises. Set XL.

Find the greatest common measure of—

1. 221, 303.	2. 361, 437.
3. 189, 448.	4. 1353, 2387.
5. 1099, 1841.	6. 1897, 2282.
7. 5759, 8593.	8. 3073, 22483.
9. 8772, 1450.	10. 16307, 32522.

80. If we leave out the quotients and omit to rewrite the divisors when in the succeeding step they are made dividends, the above process of finding the greatest common measure of two numbers may be still more concisely arranged. Thus in the cases before dealt with we should have—

221	299
156	221
65	78
65	65
G. C. M. = 13	

3528	40488
3360	3528
G. C. M. = 168	5208
	3528
	1680
	1680

87	101
84	87
3	14
2	12
G. C. M. = 1	2
	2

81. If in the process a prime factor of any divisor or dividend be evident, it may be struck out. Should it not be common to both divisor and dividend, it will then of course be entirely neglected; otherwise, it is a prime factor of the greatest common measure, and must be set aside as such.

82. The greatest common measure of three or more numbers may be found by finding first the greatest common measure of two of them, then the greatest common measure of this result and the third, and so on.

Exercises. Set XLI.

Find the greatest common measure of—

1. 2873, 4862.	2. 812, 1554.
3. 18829, 27892.	4. 25116, 40755.
5. 182, 494, 1170.	6. 1870, 1700, 2125, 3587.

83. Besides the *greatest* number *which is contained* an exact number of times in each of a set of integers, it is important to be able to find also the *least* number *which contains* each one of the set an exact number of times. The former in the case of the numbers 4, 6, 8 we know to be 2; and the latter may be seen to be 24, for 24 contains 4, 6, 8 each an exact number of times, and the same cannot be said of any smaller number.

84. A number which contains another an exact number of times is called a *multiple* of that other; for example, since 21 contains 7 exactly 3 times, 21 is said to be a multiple of 7, so 15 is a multiple of 3, and 12 of 4.

Recalling the use of the word *measure* we thus see that if one of two numbers be a multiple of the other, the latter

is a measure of the former, and *vice versâ*; that is, the words *multiple* and *measure* are correlative; 21 being a multiple of 7, 7 is a measure of 21; and 3 being a measure of 15, 15 is a multiple of 3. On account of this relation and to be suggestive of it, the term "*sub-multiple*" has sometimes been used for *measure*.

85. Exactly similar to the use of the expression "*common* measure" is the use of the expression "*common* multiple." 2 being a measure of 4, of 6, and of 8, is called a common measure of 4, 6, and 8; so 24 being a multiple of 4, of 6, and of 8, is called a common multiple of 4, 6, and 8. Further, there being other common multiples of 4, 6, and 8, such as 48, 72, &c., but none less than 24, 24 is naturally named the *least* common multiple of the numbers in question, and often, for shortness' sake, their L. C. M.

86. When the numbers whose least common multiple is wanted can be easily resolved into their prime factors, very little work is necessary.

Example. Find the least common multiple of 4, 6, 12, 15, 30, 20, 70, 18.

$$\begin{aligned}
12 &= 2 \times 2 \times 3 \\
30 &= 2 \times 3 \times 5 \\
20 &= 2 \times 2 \times 5 \\
70 &= 2 \times 5 \times 7 \\
18 &= 2 \times 3 \times 3 \\
\therefore \quad \text{L. C. M.} &= 2 \times 2 \times 3 \times 5 \times 7 \times 3 \\
&= 1260.
\end{aligned}$$

The numbers 4 and 6 are left out of consideration, since if we find a number which contains 12 we are sure that it will also contain 4 and 6; similarly 15 is neglected on account of the presence of 30. Of the factors 2, 2, 3, 5, 7, 3, taken to form the least common multiple, the $2 \times 2 \times 3$ is introduced in order that the number we are forming may contain 12; that it may contain 30 we introduce only 5, the other prime factors of 30, viz. 2, 3, being already put in; that it may contain 20, nothing further requires to be done, the prime factors of 20, viz. 2, 2, 5, being already intro-

duced; that it may contain 70 we introduce 7; and that it may contain 18 we introduce a second 3.

The resulting number found in this way clearly contains each of the given numbers, that is, is a common multiple of them; and as not one of the prime factors taken to form it can be dispensed with, it must therefore be the *least* common multiple.

Exercises. Set XLII.

Find the least common multiple of—

1. 4, 6, 15.
2. 6, 8, 10.
3. 8, 10, 12.
4. 10, 12, 14, 16.
5. 9, 12, 15, 18.
6. 15, 18, 20, 24.
7. 21, 27, 30, 36.
8. 32, 36, 40, 48.
9. 40, 50, 65, 75, 80.
10. 42, 49, 56, 63, 84.
11. 12, 20, 24, 60, 95, 100.
12. 7, 15, 21, 35, 63, 108.

87. In using the foregoing method difficulties may arise in trying to resolve the integers into their prime factors. Such resolution is not required in the method now to be explained, which besides being more generally applicable is also more expeditious.

Example 1. Find the least common multiple of 9, 12, 30, 63, 18, 49, 70.

$$\begin{array}{r|ccccccc} 12 & \cancel{9}, & 12, & 30, & 63, & 18, & 49, & 70 \\ \hline 35 & & & \cancel{5}, & 21, & \cancel{3}, & 49, & 35 \\ \hline & & & & 3, & & 7, & \end{array}$$

$$\text{L. C. M.} = 12 \times 35 \times 3 \times 7 = 8820.$$

Here we first of all strike out as before any one of the numbers which is contained an exact number of times in any of the others; then we proceed as follows to obtain one by one a series of numbers whose product must be the number sought. First we set aside 12, which, being one of the given numbers, must consequently appear as a factor of the required number. But in doing this we ensure also that the 6 which occurs as a factor of the next number, 30, will be represented in the number we are finding, so that provision has only to be made for the other factor, 5, and on this account the 5 is taken note of. Similarly under 63

we write 21, the other factor, 3, having been provided for; and so on. In other words, we divide 30 by the greatest common measure of 12 and 30, we divide 63 by the greatest common measure of 12 and 63, and so with the others. The numbers 5, 21, 3, 49, 35 are now treated like those of the previous row; but the numbers of the next row being mutually prime, there is no necessity for proceeding farther; and multiplying together the numbers found, viz. 12, 35, 3, 7, we have the number required.

Example 2. Find the least common multiple of 32, 40, 48, 6, 21, 10, and of 57, 95, 20.

$$\begin{array}{r|l} 48 & 32,\ 40,\ 48,\ \cancel{6},\ 21,\ \cancel{10} \\ \hline & 2,\ \ 5,\ \qquad\ 7, \end{array} \qquad\qquad \begin{array}{r|l} 95 & 57,\ 95,\ 20 \\ \hline & 3 \qquad 4 \end{array}$$

$$\text{L. C. M.} = 48 \times 2 \times 5 \times 7 = 3360. \qquad\qquad \text{L. C. M.} = 95 \times 3 \times 4 = 1140.$$

EXERCISES. SET XLIII.

Find the least common multiple of—

1. 2, 3, 4, 5, 6, 7, 8, 9.
2. 6, 9, 12, 15, 21, 24.
3. 10, 12, 14, 16, 18.
4. 22, 26, 33, 39, 12.
5. 3, 5, 7, 11, 13, 57.
6. 54, 81, 117, 104, 169.
7. 407, 481, 65.
8. 685, 959, 49.
9. 1127, 69, 3703.
10. 5627, 6289.

FRACTIONS.

88. The ideas conveyed by such terms as "*a half*," "*a third*," "*a quarter*," &c., are usually acquired long before entering formally on the study of arithmetic. All that is previously necessary is an acquaintance with the ideas "*two*," "*three*," "*four*," &c., and the conception of the breaking up of a naturally existing whole into equal parts. For example, if we are familiar with the idea "*six*," and see

an apple cut into equal portions which we find to be six in number, on thinking of one portion in its relation to the whole apple we seize the idea of "*a sixth of an apple*," and thereafter the abstract idea "*a sixth*." As any number less than six of these portions would form only part of the original whole, the terms "*a sixth*," "*two-sixths*," "*three-sixths*," "*four-sixths*," "*five-sixths*" are called FRACTIONS, or FRACTIONAL NUMBERS—that is, numbers less than unity. For the present we confine ourselves to fractions like the above, which are expressible as so many times some particular aliquot part of unity.

89. Fractions thus arise in connection with the actual separation of naturally existing units into equal parts. But their great value, like that of numbers in general, is in connection with arbitrarily chosen units of measurement.

Let us recur to the example of measuring a piece of wood. Suppose we have fixed upon a unit of length, say the foot, but on applying it have found that it is not contained an exact number of times in the piece to be measured—more than five times, for example, but less than six. This of course may be given as the result of our measurement, but it obviously fails in point of exactness. To overcome the difficulty one way of proceeding has already been pointed out, viz., adopting a second unit shorter than the former—the inch—finding how often it is contained in the part of the wood remaining after the measurement of the five feet, using in the same way a third unit if there remain a portion less than an inch still to be measured, and so on until we attain the degree of accuracy we may require, the result of the measurement being expressed in terms of several units. The use of fractions, however, affords us a second mode of expressing the length with exactness. The part of the wood remaining after the measurement of the five feet, although too short to be measured by the foot, may contain an *aliquot part* of the foot an exact number of times. That is to say, by trying one after another the suc-

cessively smaller aliquot parts of a foot, the half, the third, the quarter, &c., we may at last find one which is either contained an exact number of times in the portion to be measured, or sufficiently nearly so for the degree of accuracy aimed at. For example, the eighth of a foot may be contained exactly seven times, and if so, the length of the whole piece is expressed with perfect exactness in the form "*five feet and seven-eighths.*" Here, instead of several units explicitly mentioned, we have only the *foot.*

90. The point of difference between these two modes of expressing the measure of any magnitude is that, after using one unit of measurement, we try in the one case to find such an aliquot part of this unit (*e.g.*, the *eighth*, in the above) as is contained an exact number of times in the portion remaining to be measured; while in the other case we use at once a part of it already fixed and named (the *inch*), without regard to whether it be contained an exact number of times in the part to be measured or not, so that, if it be not, a third still smaller unit must be used, and so on.

The former may be called the *one-unit* system, although there are in reality two units (the *foot* and *eighth-of-a-foot*, in the above), the latter the *several-unit* system. The one-unit system will be easily seen to be the more perfect, because in the other we may soon exhaust all the fixed units at our disposal without attaining the necessary exactness, in which case fractions must after all be called into use. Thus we continually meet such expressions as *two feet three inches and five-eighths*, *five minutes fifteen seconds and three-fourths*, &c.

91. The NOMENCLATURE of fractions is very simple. The name given to each of the equal parts into which the unit is supposed to be divided is in English the ordinal adjective corresponding to the number of parts; thus, in the case of *seven* parts, each is called a *seventh.* The exceptions are *half* instead of *second*, *quarter* as well as *fourth*, and in compound words *oneth* instead of *first* (*e.g.*, twenty-

oneth). A name of this kind is called the *denomination* of the fraction, just as we speak of the denominations "inch," "foot," &c.

Evidently, in the name of every fraction there are two numbers involved. The one is the number of equal parts into which the unit is supposed to be divided, and this as implying the denomination of the fraction is called the *denominator;* the other is the number of such parts actually in the given fraction, and is called the *numerator.* For example, in the fraction "*five-sevenths,*" the denomination is *sevenths*, the denominator *seven*, and the numerator *five.*

92. NOTATION.—On looking at the fractions "three-fourths," "five-sixteenths," &c., as written at length, a short mode of writing them, which may occur to every one, is "3-4ths," "5-16ths," &c., and this form is in reality sometimes used. The mode generally adopted, however, is

$$\frac{3}{4}, \quad \frac{5}{16}, \ldots\ldots\ldots;$$

that is, a horizontal line is drawn, the numerator is written in figures above it, and the denominator below it.

When a number consists of an integer and a fraction, the symbols for the integer are first written, and then, without any connecting sign, those for the fraction; thus

$$531\tfrac{5}{16} \text{ stands for } 531 + \tfrac{5}{16}.$$

93. INTEGERS IN FRACTIONAL FORM.—If the unit be divided into three parts, one of these is expressed by the fraction "*a third,*" two of them put together form the fraction "*two-thirds,*" but three of them constitute the original unit, and cannot be spoken of as a fraction. The expression "*three-thirds*" has the *form* of a fraction, but is in reality the integer ONE. Similarly *six-thirds* is the integer TWO in fractional form, *nine-thirds* is the integer THREE, and so on. It is thus clear that all integers can be expressed in the form of fractions.

But, further, the integer ONE, besides being *three-thirds,*

is also *two-halves*, *four-fourths*, *five-fifths*, &c., and thus we see that there is an infinite variety of ways in which an integer can be expressed in fractional form.

Example 1. Change *six* into fractional form with the denomination *tenths*.

ONE is *ten-tenths*,
therefore SIX is *sixty-tenths*.

Example 2. Express 19 in fractional form with the denominators 4 and 10.

$$1=\frac{4}{4} \quad \therefore \quad 19=\frac{4\times 19}{4}=\frac{76}{4},$$

$$\text{and } 1=\frac{10}{10} \quad \therefore \quad 19=\frac{10\times 19}{10}=\frac{190}{10}.$$

94. Further, it is evident from this that a number which is partly integral and partly fractional may be expressed wholly in fractional form.

Example 1. Express *three and two-thirds* wholly in fractional form.

THREE is *nine-thirds*,
therefore THREE and *two-thirds* is *eleven-thirds*.

Example 2. Express $6\frac{7}{12}$ and $13\frac{3}{100}$ wholly in fractional form.

$$6\frac{7}{12}=6+\frac{7}{12}=\frac{72}{12}+\frac{7}{12}=\frac{79}{12}.$$

$$13\frac{3}{100}=\frac{1300}{100}+\frac{3}{100}=\frac{1303}{100}.$$

EXERCISES. SET XLIV.

1. Express 2 ft. 6 in. in terms of the *foot* alone; 1s. 1d. in terms of the *shilling* alone; and £1 7s. 6d. in terms of the *pound* alone.

2. Write in words and in figures the fraction obtained by dividing a whole into ten parts and taking seven of them.

3. Write in figures *fifty hundredths*, *a hundred and one thousandths*, *a hundred and one thousandth*, *thirteen ten-millionths*, *twenty-one hundred-millionths*.

4. Express 5 in fractional form with the denomination *fifths*, *tenths*, *hundredths*, *thousandths*.

5. Express 2, 7, 13, 20 in the form of fractions—(1) with 4 for denominator, (2) with 48 for denominator.

6. Convert 3, 13, 23, 103 into the form of fractions—(1) with denominator 1, (2) with denominator 144.

Express each of the following numbers wholly in fractional form :—

7. $3\frac{1}{8}$, $30\frac{1}{8}$, $2\frac{4}{5}$, $200\frac{4}{5}$, $2000\frac{14}{15}$.

8. $2\frac{1}{10}$, $14\frac{33}{100}$, $17\frac{841}{1000}$, $1008\frac{2619}{10000}$.

9. $17\frac{3}{100}$, $18\frac{3}{1000}$, $24\frac{27}{100000}$, $34000\frac{13}{100000}$.

10. $26\frac{98}{99}$, $26\frac{97}{98}$, $112\frac{118}{199}$, $347\frac{347}{999}$, $101\frac{29}{101}$.

95. Although it may be convenient at times to express in fractional form numbers which are not fractions, it must be remembered that the change is made merely for a temporary purpose, and that otherwise the fractional form is never used unless for real fractions, just as we occasionally find it serviceable to express in the unusual form of pence or farthings a sum of money which amounts to several pounds. Now unity being equal to two-halves, three-thirds, four-fourths, &c., it must before this have been evident that in any case where the numerator is not less than the denominator there is an integer in disguise. We are therefore now prepared to detect such sham fractions, and to express them in their proper form.

Example 1. Express *thirteen-fifths* in its proper form.

Thirteen-fifths is the sum of *five-fifths* and *five-fifths* and *three-fifths*; but *five-fifths* is ONE, consequently *thirteen-fifths* is the same as TWO and *three-fifths*.

Here our aim was to ascertain how many times five-fifths are contained in thirteen-fifths, that is, how many times unity is contained in the given number; and such being the object in every case, division of numerator by denominator clearly gives all that is required.

Example 2. Express $\frac{25}{8}$ and $\frac{36}{18}$ in their proper form.

$$\frac{25}{8}=\frac{8}{8}+\frac{8}{8}+\frac{8}{8}+\frac{1}{8},$$

$$=1+1+1+\frac{1}{8},$$

$$=3\tfrac{1}{8}.$$

$$\frac{36}{18}=2.$$

EXERCISES. SET XLV.

Express the following numbers in their usual form :—

1. $\frac{3}{2}$, $\frac{10}{3}$, $\frac{12}{12}$, $\frac{19}{8}$, $\frac{33}{13}$, $\frac{55}{39}$, $\frac{91}{13}$.

2. $\frac{113}{10}$, $\frac{667}{100}$, $\frac{41769}{1000}$, $\frac{30007}{1000}$, $\frac{342}{31}$, $\frac{499}{99}$.

3. $\frac{246}{7}$, $\frac{3265}{11}$, $\frac{28192}{9}$, $\frac{948}{237}$, $\frac{1134}{126}$, $\frac{8419}{30}$.

4. $3\frac{8}{9}$, $7\frac{14}{6}$, $198\frac{188}{100}$, $4\frac{44}{16}$, $1\frac{116}{15}$, $109\frac{8808}{22}$.

96. It has been already remarked that if we multiply a number by another, and divide the product by a third number, we obtain the same final result as we do when the operations are reversed. There is a particular case of this which, under a different form, now deserves special attention. When we divide a number by 8, and multiply the quotient by 3, the result is three-eighths of the given number. When, on the other hand, we multiply by 3, and divide the product by 8, the result is the eighth part of three times the given number. But the two results being the same, we see that three-eighths of a number is the same as the eighth part of three times the number; and if the number be unity, this amounts to saying that *three-eighths is the same as the eighth part of three*, or, in symbols, $\frac{3}{8} = 3 \div 8$. With the truth of this we are quite familiar, and would act upon it unhesitatingly. For example, if we were asked to divide 3 equal-sized apples among 8 boys, that is, to give each boy the eighth part of 3 apples, we should doubtless do so by giving each boy an eighth part of each of the apples, that is, three-eighths of one of them.

A noticeable result of this is that the symbol of division, $\div$, is to a great extent supplanted by the fractional notation. The quotient of 19 by 7, which is written $19 \div 7$, being the seventh part of 19, and this being the same as nineteen-sevenths, which is written $\frac{19}{7}$, it clearly follows that we may use with much convenience $\frac{19}{7}$ for $19 \div 7$. And this change of notation is not confined to the case

where the divisor and dividend are integers, but is perfectly general—*e.g.*,

$$\frac{3\frac{1}{2}}{\frac{7}{8}} \text{ is used for } 3\tfrac{1}{2}\div\tfrac{7}{8}$$

$$\frac{\frac{1}{2}-\frac{1}{3}}{2\frac{1}{2}+\frac{1}{7}} \text{ is used for } (\tfrac{1}{2}-\tfrac{1}{3})\div(2\tfrac{1}{2}+\tfrac{1}{7}),\ \&c.,$$

the names *numerator* and *denominator* being extended so as to be synonymous with *dividend* and *divisor*. In this usage, however, unless care be taken, ambiguity may arise; thus $\frac{\frac{2}{3}}{5}$ may mean either $\frac{2}{3}\div 5$ or $3\div\frac{2}{5}$. To avoid this, all that is necessary is to make the line separating any dividend and divisor more conspicuous than any line of the same nature occurring already in the dividend or divisor—*e.g.*,

$$3\div\tfrac{2}{5} \text{ is written } \cfrac{3}{\;\frac{2}{5}\;} \text{ or } \frac{3}{\frac{2}{5}},$$

$$\tfrac{2}{3}\div 5 \text{ is written } \cfrac{\;\frac{2}{3}\;}{5} \text{ or } \frac{\frac{2}{3}}{5},\ \&c.$$

Further, this explains the mode shown at page 31 of dealing with the remainder obtained on dividing an integer by an integer.

97. We have seen that any integer is expressible in fractional form in an infinite variety of ways; the same, however, is true of any fraction also. For example, the forms *three-fourths*, *six-eighths*, *nine-twelfths*, &c., all express the same fractional number.

That *three-fourths* is the same as *six-eighths* is easily made clear. When a *fourth* of any whole is divided into *two* equal parts, each of them must be an *eighth* of the whole; and one-fourth thus being two-eighths, three-fourths must be six-eighths; in symbols, $\frac{3}{4} = \frac{6}{8}$. Similarly we may show that $\frac{3}{4} = \frac{9}{12}$, or $\frac{12}{16}$, or $\frac{15}{20}$, ...; and, generally, it is true that *if the numerator and denominator of a fraction be both multiplied by the same number the magnitude of the fraction is unaltered.* For, taking as an example the multiplier 7, each of the parts of the second fraction is 7 times less than each

part of the first, but there are 7 times more parts taken in the second than in the first; *e.g.*, $\frac{2}{5}$ and $\frac{14}{35}$.

Remembering what has just been said regarding the sign of division, we see that we have had already to deal with this truth under another form, viz., *if divisor and dividend be both multiplied by the same number the quotient is unaltered.*

Example 1. Express *nine-tenths* in other fractional forms.

One-tenth is *two-twentieths;* therefore *nine-tenths* are *eighteen-twentieths.* Similarly, *nine-tenths* are *twenty-seven thirtieths*, &c.

Example 2. Express $\frac{5}{8}$ and $\frac{5}{12}$ as fractions with the denominator 24.

The denominator 8 must be multiplied by 3 to produce 24, and, in order to preserve unaltered the value of the fraction, the numerator also must be multiplied by 3. Thus—

$$\frac{5}{8}=\frac{5\times 3}{8\times 3}=\frac{15}{24}.$$

Similarly, $$\frac{5}{12}=\frac{5\times 2}{12\times 2}=\frac{10}{24}.$$

EXERCISES. SET XLVI.

1. Write in words and in figures the fraction of a florin obtained by taking the hundredth part of seventeen florins.
2. Express $\frac{2}{3}$ as a fraction with the denominators 9, 12, 27, 300.
3. Express $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{12}$ as fractions with the denominator 24.
4. Express $\frac{2}{3}$, $\frac{3}{5}$, $\frac{1}{6}$, $\frac{7}{10}$, $\frac{2}{15}$ as fractions with the denominator 30.
5. Express $\frac{2}{7}$, $\frac{3}{5}$, $\frac{11}{14}$, $\frac{12}{35}$, $\frac{1}{2}$ as fractions with the denominator 70.
6. Express $\frac{3}{5}$ as a fraction with the numerators 9, 15, 72.
7. Express $\frac{2}{7}$, $\frac{3}{5}$, $\frac{6}{11}$, $\frac{9}{10}$ as fractions with the numerator 54.
8. Express $\frac{1}{2}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{8}$, $\frac{7}{25}$, $\frac{3}{16}$ as fractions with the lowest possible power of 10 for denominator.

98. Conversely, it is clear that if the numerator and denominator have a common factor, we may divide both by this factor without altering the value of the fraction. For example, observing that 27 and 72 have the common factor 9, we conclude that

$$\frac{27}{72}=\frac{3}{8}.$$

A fractional form derivable in this way from another is, for an evident reason, said to be simpler than that other; and when division like this is no longer possible, *i.e.*, when numerator and denominator are mutually prime, the fraction is said to be in its *simplest form*.

As fractions are always spoken of and used in their simplest forms, unless for some temporary purpose, practice in deriving the simplest from any other form is desirable.

Example 1. Express $\frac{32}{72}$ and $\frac{396}{693}$ in their simplest forms.

$$\frac{32}{72}=\frac{4\times 8}{9\times 8}=\frac{4}{9}.$$

$$\text{And}\quad \frac{396}{693}=\frac{36\times 11}{63\times 11}=\frac{36}{63}=\frac{4\times 9}{7\times 9}=\frac{4}{7}.$$

If no factor common to numerator and denominator be apparent, and we cannot at the same time affirm with certainty that they are mutually prime, we must have recourse to the known process of finding their greatest common measure.

Example 2. Change $\frac{2405}{5135}$ to its simplest form.

One common factor of numerator and denominator is here apparent, viz., 5, and we thus have

$$\frac{2405}{5135}=\frac{481}{1027}.$$

But now, as we may not be able to tell whether 481 and 1027 have a common factor or not, we apply the testing process, and find their greatest common measure to be 13. This at once leads to the required simplest form, viz.,

$$\frac{37}{79}.$$

The simplest form of a fraction is naturally that from which we can acquire most easily a definite idea of the magnitude of the fraction. Thus, we may have some idea of the magnitude of the number specified in the form $\frac{27}{72}$, but the idea is more easily obtained from considering the form $\frac{9}{24}$, and still more so by thinking of the *simplest* form $\frac{3}{8}$.

When a fraction, unlike this example, has in its simplest form a large numerator and denominator, an idea of its magnitude may be got by considering simpler forms which are *approximately* equal to it. The mode of finding these is explained farther on (§ 104, &c.).

EXERCISES. SET XLVII.

Express each of the following fractions in its simplest form :—

1. $\frac{48}{120}, \frac{30}{135}, \frac{180}{315}$.
2. $\frac{57}{72}, \frac{141}{162}, \frac{252}{261}$.
3. $\frac{117}{135}, \frac{936}{1008}, \frac{972}{1053}$.
4. $\frac{121}{297}, \frac{352}{880}, \frac{319}{429}$.
5. $\frac{91}{221}, \frac{112}{133}, \frac{153}{187}$.
6. $\frac{3130}{4695}, \frac{1029}{1176}, \frac{2853}{3170}$.
7. $\frac{215}{399}, \frac{1944}{2817}, \frac{1397}{2255}$.
8. $\frac{987}{31426}, \frac{4627}{9198}, \frac{3465}{4851}$.

99. When we have two fractions of the same denomination, *two-fifths* and *three-fifths*, we can at once tell, as in similar cases already seen, which is the greater, the sum of the two, and their difference. In the case of fractions of different denominations, such as *two-fifths* and *three-fourths*, this is not directly possible. We have learned, however, that a fraction can be expressed in an infinite variety of denominations; that *two-fifths* is expressible as a fraction with a denominator which is any multiple of 5, *e.g.*, *four-tenths*, &c.; and *three-fourths* as a fraction with a denominator which is any multiple of 4, *e.g.*, *six-eighths*, &c. Now 20 is a multiple of both 5 and 4, and thus *two-fifths* and *three-fourths* can each be expressed as so many *twentieths*, and this being done, comparison, addition, and subtraction are possible. Moreover, as there is an infinite number of multiples of 5 and 4 besides 20, we see that the given fractions can be expressed in the same denomination in an infinite variety of ways. The *lowest* multiple, however, is clearly preferable.

Example 1. Express $\frac{5}{8}$ and $\frac{7}{12}$ as fractions of the same denomination.

48 is a common multiple of 8 and 12, and therefore the given fractions may be expressed as *forty-eighths*, viz.,

$$\frac{5}{8}=\frac{30}{48},$$

and $$\frac{7}{12}=\frac{28}{48}.$$

48, however, is not the *lowest* common multiple, but 24; and thus,

$$\frac{5}{8}=\frac{15}{24},$$

and $$\frac{7}{12}=\frac{14}{24}.$$

Example 2. Express in the simplest way $\frac{3}{8}$, $\frac{6}{9}$, $\frac{4}{15}$ as fractions of the same denomination.

The lowest common multiple of 8, 9, 15 is 360; therefore each of the fractions can be expressed in 360ths. In order they are $\frac{135}{360}$, $\frac{240}{360}$, $\frac{96}{360}$. It will be found, however, that these may be put in the simpler forms $\frac{45}{120}$, $\frac{80}{120}$, $\frac{32}{120}$; so that 360ths is not the lowest denomination in which the fractions can be expressed. The cause of this is that we omitted to notice that one of the given fractions ($\frac{6}{9}$) is not in its simplest form.

It is worthy of remark that if we wish only to compare the magnitudes of the fractions, the expressing of them as fractions with the same *numerator* is equally effective. Thus *two-fifths* and *three-fourths*, being equal respectively to *six-fifteenths* and *six-eighths*, we see that the latter is the greater because an *eighth* is greater than a *fifteenth*, just as we know that *six guineas* are more than *six pounds*.

EXERCISES. SET XLVIII.

Express as fractions with the lowest possible common denominator—

1. $\frac{3}{4}, \frac{1}{2}, \frac{5}{8}, \frac{7}{12}.$ 2. $\frac{3}{5}, \frac{2}{15}, \frac{7}{10}, \frac{5}{6}.$ 3. $\frac{3}{8}, \frac{5}{12}, \frac{7}{16}, \frac{11}{20}.$

4. $\frac{4}{7}, \frac{8}{21}, \frac{9}{35}, \frac{13}{28}, \frac{4}{49}.$ 5. $\frac{2}{11}, \frac{7}{15}, \frac{8}{55}, \frac{5}{44}, \frac{3}{20}.$

6. $\frac{3}{14}, \frac{5}{18}, \frac{8}{63}, \frac{14}{45}, \frac{29}{35}.$ 7. $\frac{8}{15}, \frac{4}{39}, \frac{18}{65}, \frac{7}{9}, \frac{9}{25}.$

8. $\frac{7}{24}, \frac{17}{30}, \frac{8}{45}, \frac{17}{54}, \frac{19}{48}.$

9. $\frac{13}{27}, \frac{11}{36}, \frac{11}{12}, \frac{55}{108}, \frac{2}{81}.$

10. $\frac{3}{11}, \frac{4}{187}, \frac{9}{34}, \frac{53}{85}, \frac{1}{255}, \frac{2}{51}.$

11. $\frac{4}{19}, \frac{2}{3}, \frac{31}{171}, \frac{13}{114}, \frac{11}{76}, \frac{4}{45}.$

Arrange each of the following sets of numbers in ascending order of magnitude :—

12. $\frac{8}{5}, \frac{5}{7}, \frac{6}{11}, \frac{24}{35}.$

13. $\frac{13}{49}, \frac{2}{7}, \frac{9}{35}, \frac{3}{11}.$

14. $\frac{13}{30}, \frac{67}{154}, \frac{100}{231}, \frac{167}{385}.$

15. $\frac{3^4}{2^5}, 2\frac{6}{11}, \frac{632}{253}, \frac{57}{23}.$

100. Addition.—After what has been said incidentally in the preceding paragraph, little explanation on this head or the next is necessary.

Example 1. Find the sum of *three-fifths* and *two-thirds*.

Three-fifths is the same as *nine-fifteenths*, and *two-thirds* is the same as *ten-fifteenths;* therefore the required sum is the sum of *nine-fifteenths* and *ten-fifteenths*, that is, *nineteen-fifteenths*, or ONE and *four-fifteenths*.

Example 2. Find the sum of $\frac{2}{7}$, $\frac{5}{8}$, $\frac{3}{4}$, and $\frac{23}{28}$.

$$\frac{2}{7}+\frac{5}{8}+\frac{3}{4}+\frac{23}{28}=\frac{16}{56}+\frac{35}{56}+\frac{42}{56}+\frac{46}{56}$$
$$=\frac{16+35+42+46}{56}$$
$$=\frac{139}{56}$$
$$=2\tfrac{27}{56}.$$

Example 3. Simplify the expression $3\frac{1}{4}+\frac{2}{5}+100\frac{4}{15}+12\frac{3}{10}$.

$$\text{Given expression }=3+100+12+\frac{1}{4}+\frac{2}{5}+\frac{4}{15}+\frac{3}{10}$$
$$=115+\frac{15+24+16+18}{60}$$
$$=115+\frac{73}{60}$$
$$=116\tfrac{13}{60}.$$

If, in the items, integers occur in the form of fractions, there is some little advantage in changing them at once into their proper form.

EXERCISES. SET XLIX.

Perform the following additions :—

1. $\frac{5}{12}+\frac{7}{12}+\frac{1}{12}+\frac{11}{12}$.

2. $\frac{1}{2}+\frac{3}{4}+\frac{5}{8}+\frac{3}{16}$.

3. $\frac{3}{5}+\frac{2}{15}+\frac{7}{20}+\frac{7}{10}$.

4. $\frac{1071}{10000}+\frac{9}{100}+\frac{3}{10}+\frac{117}{1000}$.

5. $9\frac{7}{10}+90\frac{7}{10^2}+900\frac{7}{10^3}$.

6. $\frac{4}{9}+\frac{11}{15}+\frac{5}{18}+\frac{16}{45}$.

7. $\frac{5}{21}+\frac{3}{7}+\frac{18}{35}+\frac{7}{10}$.

8. $\frac{11}{14}+\frac{5}{21}+\frac{11}{20}+\frac{11}{30}$.

9. $6\frac{1}{2}+\frac{3}{11}+2\frac{10}{33}+\frac{5}{6}$.

10. $\frac{7}{8}+10\frac{1}{10}+7+\frac{11}{60}+\frac{11}{12}$.

11. $112\frac{11}{36}+6\frac{1}{27}+10\frac{7}{24}+5\frac{7}{18}$.

12. $\frac{1}{5}+19\frac{1}{7}+3\frac{7}{12}+\frac{1}{84}+16\frac{11}{42}$.

13. $2\frac{1}{33}+\frac{3}{7}+\frac{3}{22}+101\frac{17}{42}+\frac{1}{18}$.

14. $3\frac{1}{15}+2\frac{1}{4}+6\frac{7}{30}+\frac{7}{18}+4\frac{11}{21}$.

15. $2\frac{1}{22}+10\frac{3}{8}+\frac{5}{6}+1\frac{1}{15}+\frac{4}{33}$.

16. $\frac{72}{143}+2\frac{1}{26}+\frac{5}{91}+5\frac{3}{77}$.

17. $\frac{15}{119}+\frac{19}{180}+\frac{13}{340}+19\frac{48}{85}$.

18. $\frac{31}{105}+13\frac{20}{143}+\frac{17}{42}+\frac{37}{165}$.

19. $\frac{1}{24}+\frac{2^2}{3^2}+\frac{3^3}{4^3}+\frac{29}{9^2}+\frac{33}{2^3}+\frac{17}{8^2}$.

20. $\frac{4^4}{5^4}+\frac{6}{5^3}+\frac{12^2}{10^2}+5\frac{1}{20}+\frac{12}{25^2}+\frac{11}{20^2}$.

101. SUBTRACTION.—Example 1. What is the difference between *three-sevenths* and *one-half?*

Three-sevenths is the same as *six-fourteenths*, and *one-half* is the same as *seven-fourteenths;* consequently the required difference is the difference between *six-fourteenths* and *seven-fourteenths*, that is, *one-fourteenth*.

Example 2. Subtract $\frac{3}{10}$ from $\frac{19}{35}$.

$$\frac{19}{35}-\frac{3}{10}=\frac{38}{70}-\frac{21}{70}=\frac{17}{70}.$$

Example 3. From 16 take $2\frac{7}{15}$.

$$16-2\tfrac{7}{15}=15\tfrac{15}{15}-2\tfrac{7}{15}=13\tfrac{8}{15}.$$

Example 4. From $4\frac{1}{3}$ take $2\frac{4}{7}$.

$$\begin{aligned}4\tfrac{1}{3}-2\tfrac{4}{7}&=\tfrac{1}{3}+4-2\tfrac{4}{7}\\&=\tfrac{1}{3}+1\tfrac{3}{7}\quad=\tfrac{7}{21}+1\tfrac{9}{21}\\&=1\tfrac{16}{21}.\end{aligned}$$

Or, $$4\tfrac{1}{5}-2\tfrac{2}{7}=4\tfrac{7}{35}-2\tfrac{10}{35}=3\tfrac{42}{35}-2\tfrac{10}{35}$$
$$=1\tfrac{32}{35}.$$

Example 5. Perform the operations indicated in the expression $3\frac{1}{4}+2\frac{3}{7}-(1\frac{3}{8}-\frac{2}{7}+\frac{1}{2})-3\frac{1}{3}$.

Given expression $=3\frac{42}{168}+2\frac{72}{168}-(1\frac{63}{168}-\frac{48}{168}+\frac{84}{168})-3\frac{56}{168}$

$=5\frac{114}{168}-1\frac{99}{168}-3\frac{56}{168}$

$=4\frac{15}{168}-3\frac{56}{168}$

$=\dfrac{127}{168}.$

Exercises. Set L.

Perform the operations indicated in the following expressions:—

1. $\dfrac{12}{19}-\dfrac{4}{19}$, $\dfrac{11}{18}-\dfrac{5}{18}$, $\dfrac{31}{64}-\dfrac{19}{64}$, $\dfrac{79}{100}-\dfrac{54}{100}$.
2. $2\frac{13}{25}-1\frac{8}{25}$, $12\frac{9}{100}-8\frac{1}{20}$, $5\frac{23}{100}-2\frac{161}{1000}$, $32\frac{1}{2}-17\frac{1}{4}$.
3. $\dfrac{3}{5}-\dfrac{2}{7}$, $\dfrac{4}{15}-\dfrac{2}{9}$, $\dfrac{17}{35}-\dfrac{3}{49}$, $\dfrac{17}{65}-\dfrac{10}{39}$.
4. $\dfrac{17}{108}-\dfrac{7}{96}$, $\dfrac{227}{1000}-\dfrac{131}{680}$, $\dfrac{137}{700}-\dfrac{209}{1820}$, $\dfrac{11}{720}-\dfrac{11}{1728}$.
5. $12-7\frac{1}{2}$, $20-11\frac{4}{7}$, $35-9\frac{14}{17}$, $12-11\frac{144}{201}$.
6. $13\frac{1}{2}-10\frac{3}{4}$, $13\frac{1}{3}-4\frac{6}{7}$, $2\frac{5}{12}-1\frac{16}{21}$, $100\frac{11}{84}-9\frac{77}{72}$.
7. $100\frac{37}{1000}-90\frac{248}{840}$, $8\frac{1}{2}-2\frac{2}{3}-2\frac{1}{6}$, $12\frac{1}{2}-7\frac{7}{9}-3\frac{1}{24}$.
8. $15-3\frac{1}{4}-2\frac{4}{7}$, $16\frac{2}{3}-11\frac{11}{18}-1\frac{13}{18}$, $10\frac{1}{3}-(7\frac{3}{4}-2\frac{9}{10})$.
9. $1-\dfrac{1}{2^2}-\dfrac{1}{3^2}-\dfrac{1}{4^2}$, $\dfrac{13}{14}-\left(14\frac{7}{24}-13\frac{35}{36}-\frac{1}{12}-\frac{1}{12}\right)$.
10. $3\frac{2}{3}-\left(4\frac{1}{4}-2\frac{2}{3}-\dfrac{5}{6}\right)-\left(\dfrac{23}{24}-\dfrac{7}{18}-\dfrac{1}{6}-\dfrac{1}{3}\right)$.
11. $\left(12\frac{1}{10}-10\frac{1}{12}-\dfrac{3}{25}\right)+\left(6\frac{1}{3}-3\frac{9}{100}-2\frac{1}{5}\right)+\left(1\frac{11}{24}-\dfrac{7}{8}-\dfrac{1}{16}\right)$.
12. $\left(13\frac{1}{4}+2\frac{1}{3}-2\frac{1}{6}\right)-\left(7\frac{1}{2}-6\frac{5}{8}+\dfrac{1}{12}\right)-\left(3\frac{1}{3}-1\frac{3}{4}-\dfrac{23}{24}\right)$.

102. Multiplication.—Multiplication of a fraction by an integer presents no new feature, and therefore no difficulty.

Example 1. Multiply $\frac{4}{7}$ by 4 and $\frac{3}{8}$ by 5.

(1.) $$\frac{4}{5}\times 4 \text{ means } \frac{4}{5}+\frac{4}{5}+\frac{4}{5}+\frac{4}{5},$$

and this sum $=\frac{16}{5}$ or $3\frac{1}{5}$.

(2.) $$\frac{8}{9}\times 5=\frac{8\times 5}{9}=\frac{40}{9}=4\frac{4}{9}.$$

When, however, the multiplier is a fraction, a little more consideration is necessary to obtain the product. If, for instance, the multiplier in the above example were $\frac{2}{3}$ instead of 4, this we know means that instead of taking *four times* the multiplicand we are to take only *two-thirds* of it. Our object then is to find *twice the third part of four-fifths*. Now the third part of *one*-fifth is a fifteenth, therefore the third part of *four*-fifths is four-fifteenths, and consequently *twice* the third part of four-fifths is eight-fifteenths. Thus—

$$\frac{4}{5}\times\frac{2}{3}=\frac{4\times 2}{5\times 3}=\frac{8}{15}.$$

Example 2. Multiply 4 by $\frac{3}{5}$.

The fifth part of 4 is $\frac{4}{5}$, therefore *three* fifth parts of 4 are the same as 3 times $\frac{4}{5}$, that is $\frac{12}{5}$, or $2\frac{2}{5}$.

Example 3. Find the product of $\frac{4}{35}$, $\frac{3}{8}$, $\frac{25}{27}$.

$$\begin{aligned}\frac{4}{35}\times\frac{3}{8}\times\frac{25}{27}&=\frac{4\times 3}{35\times 8}\times\frac{25}{27}\\&=\frac{4\times 3\times 25}{35\times 8\times 27}=\frac{300}{7560}\\&=\frac{5}{126}.\end{aligned}$$

The factors that may be struck out of numerator and denominator are generally more easily seen before performing the multiplication than after; and if removed then there is considerable saving of labour. Thus, in the above, before finding in the second line the product of 4, 3, 25, and of 35, 8, 27, we see that 4 being a common factor of 4 and 8 is a common factor of numerator and denominator, and similarly also 3 and 5. Striking these out at this stage, we have

$$\frac{1\times 1\times 5}{7\times 2\times 9},$$

and now performing the multiplication indicated there results $\frac{4}{180}$ as before.

Example 4. Multiply together $\frac{27}{49}$, $\frac{14}{15}$, and $\frac{28}{45}$.

$$\frac{27}{49}\times\frac{14}{15}\times\frac{28}{45}=\frac{27\times14\times28}{49\times15\times45}$$
$$=\frac{3\times2\times28}{7\times15\times5}=\frac{1\times2\times4}{1\times5\times5}$$
$$=\frac{8}{25}.$$

Here we first notice, perhaps, that 27 and 45 have a common factor 9, which leads us to substitute for them in the next line 3 and 5 respectively; similarly we put 2 and 7 instead of 14 and 49; and 15 and 28, having no common factor, are rewritten. The expression thus obtained is now treated like the preceding, and at length, when no factor common to numerator and denominator exists, the necessary multiplication is performed.

Factors which are partly integral and partly fractional are more easily dealt with when expressed wholly in fractional form.

Example 5. Simplify the expression $(\frac{3}{4}+2\frac{1}{5}-\frac{1}{2})\times1\frac{1}{9}$.

$$\text{Given expression}=(2+\frac{15+4-10}{20})\times1\frac{1}{9}$$
$$=2\frac{9}{20}\times1\frac{1}{9}$$
$$=\frac{49}{20}\times\frac{10}{9}=\frac{49}{18}$$
$$=2\frac{13}{18}.$$

We observe from the mode of finding the product that in the case of fractional factors, as in the case, already noticed, of integral factors, the order is immaterial; *e.g.*, $\frac{3}{7}\times\frac{4}{5}$ is the same as $\frac{4}{5}\times\frac{3}{7}$, a statement which is more astonishing when put in the form—If three-sevenths of four-fifths (of a foot, say) be taken, the result is the same as if four-fifths of three-sevenths were taken.

Any two numbers whose product is unity are said to be *reciprocals* the one of the other. Thus, $\frac{3}{5}$ and $\frac{5}{3}$ are reciprocals, 4 is the reciprocal of $\frac{1}{4}$, &c.

Finally, as a matter of notation, it is to be remarked that instead of "$\frac{3}{4} \times \frac{6}{7}$" there is in use the mixed notation "$\frac{6}{7}$ of $\frac{3}{4}$," an alternative of the same kind as if we were in the habit of writing "9 times 7" instead of "7×9."

Exercises. Set LI.

1. Multiply $\dfrac{313}{4200}$ by 3, 4, 5, 6, 7. 2. Multiply $\dfrac{541}{1980}$ by 10, 11, 12, 15.

Perform the operations indicated in the following expressions:—

3. $\dfrac{3}{4} \times \dfrac{6}{7}$, $\dfrac{2}{5} \times \dfrac{15}{24}$, $\dfrac{7}{180} \times \dfrac{48}{49}$.

4. $\dfrac{10}{27} \times \dfrac{6}{25}$, $\dfrac{21}{29} \times \dfrac{87}{91}$, $\dfrac{57}{69} \times \dfrac{46}{95}$.

5. $2\frac{1}{7} \times \dfrac{14}{25}$, $2\frac{124}{143} \times 3\frac{18}{55}$.

6. $12\frac{3}{8} \times 2\frac{2}{11}$, $2\frac{10}{33} \times 1\frac{86}{133}$.

7. $\dfrac{12}{35} \times \dfrac{5}{18} \times \dfrac{21}{22}$, $\dfrac{120}{77} \times \dfrac{165}{182} \times \dfrac{637}{1700}$.

8. $\dfrac{39}{14} \times \dfrac{51}{65} \times \dfrac{21}{85}$, $\dfrac{39}{74} \times \dfrac{82}{195} \times \dfrac{185}{240}$.

9. $\dfrac{8}{9} \times 21 \times 4\frac{1}{8}$, $\dfrac{5}{14} \times 2\frac{1}{10} \times 4$.

10. $100 \times \dfrac{6}{7} \times 4\frac{1}{6}$, $3\frac{1}{4} \times 6\frac{2}{3} \times 1\frac{3}{52}$.

11. $\dfrac{2^5}{3^3} \times (1\frac{1}{8})^2$, $\left(3\frac{1}{4} - \dfrac{7}{8}\right) \times 16$.

12. $\left(3\frac{3}{8} - 2\frac{7}{12}\right) \times \left(5 - \dfrac{7}{19}\right) \times 8\frac{1}{2}$.

13. $\left(\dfrac{1}{2} + \dfrac{3}{5}\right)^2 + \left(3\frac{1}{2} - \dfrac{1}{10} - \dfrac{1}{5}\right)^3$.

14. $2 \times (7\frac{1}{8} - 3\frac{1}{6}) - \left(\dfrac{7}{15} \times 1\frac{1}{10}\right)$.

15. $\left(1 - \dfrac{1}{3} + \dfrac{1}{5}\right)^2 \times \left(\dfrac{1}{2} - \dfrac{1}{4} \times \dfrac{1}{6}\right)$.

16. $\left(\dfrac{13}{14} \times \dfrac{7}{15} - \dfrac{1}{5}\right)^2 \times \left(\dfrac{12}{25} \times \dfrac{5}{8} - \dfrac{11}{100}\right)^2$.

103. Division.—Example 1. Divide $\frac{3}{7}$ by 4.

The quotient sought we know to be such a number that if it be taken 4 times the result is $\frac{3}{7}$; it therefore must be a fourth part of $\frac{3}{7}$, that is, $\frac{3}{7} \times \frac{1}{4}$. Hence—

$$\begin{aligned} \frac{3}{7} \div 4 &= \frac{3}{7} \times \frac{1}{4} \\ &= \frac{3}{28}. \end{aligned}$$

Example 2. Divide $\frac{5}{8}$ by $\frac{2}{3}$.

$\frac{2}{3}$ of the quotient we know to be $\frac{5}{8}$, therefore $\frac{3}{2}$ of $\frac{2}{3}$ of the quotient must be $\frac{3}{2}$ of $\frac{5}{8}$. But $\frac{3}{2}$ of $\frac{2}{3}$ of the quotient is exactly the whole of the quotient; consequently the quotient is $\frac{3}{2}$ of $\frac{5}{8}$, or $\frac{5}{8} \times \frac{3}{2}$; that is—

$$\begin{aligned} \frac{5}{8} \div \frac{2}{3} &= \frac{5}{8} \times \frac{3}{2} \\ &= \frac{15}{16}. \end{aligned}$$

Or we may reason thus:—When $\frac{5}{8} \times \frac{8}{3}$ is multiplied by $\frac{3}{8}$ the result is $\frac{5}{8} \times \frac{8}{3} \times \frac{3}{8}$, which equals $\frac{5}{8} \times 1$, or simply $\frac{5}{8}$. We have thus found a number, viz., $\frac{5}{8} \times \frac{8}{3}$, such that if it be multiplied by the given divisor, the result is the given dividend: it must therefore be the quotient sought.

We thus see that division by any number whatever resolves itself into multiplication by the reciprocal of the number.

Example 3. Simplify the expression $\dfrac{2\frac{1}{7}+3\frac{1}{4}}{5\frac{1}{8}-1\frac{5}{6}}$.

$$\begin{aligned}\text{Given expression} &= (2\tfrac{1}{7}+3\tfrac{1}{4}) \div (5\tfrac{1}{8}-1\tfrac{5}{6})\\ &= (2\tfrac{4}{28}+3\tfrac{7}{28}) \div (5\tfrac{3}{24}-1\tfrac{20}{24})\\ &= 5\tfrac{11}{28} \div 3\tfrac{7}{24} = \frac{151}{28} \div \frac{79}{24}\\ &= \frac{151}{28} \times \frac{24}{79} = \frac{151}{7} \times \frac{6}{79}\\ &= \frac{906}{553} \quad = 1\tfrac{353}{553}.\end{aligned}$$

Exercises. Set LII.

1. Divide $\frac{30}{41}$ by 5, 7, 21, 48.
2. Divide $\frac{300}{487}$ by 11, 60, 64, 140.

Perform the operations indicated in the following expressions:—

3. $12 \div \frac{3}{5}$, $16 \div \frac{14}{15}$, $9 \div \frac{12}{13}$.
4. $\frac{3}{8} \div \frac{12}{13}$, $\frac{14}{15} \div \frac{35}{48}$, $\frac{108}{125} \div \frac{81}{100}$.
5. $\frac{27}{35} \div \frac{18}{49}$, $\frac{39}{58} \div \frac{52}{87}$, $\frac{51}{55} \div \frac{68}{121}$.
6. $\frac{58}{65} \div \frac{85}{52}$, $\frac{92}{93} \div \frac{69}{62}$, $\frac{74}{82} \div \frac{31}{37}$.
7. $2\frac{1}{3} \div 1\frac{1}{9}$, $6\frac{1}{8} \div 4\frac{1}{5}$, $10\frac{6}{7} \div 1\frac{4}{21}$.
8. $6\frac{5}{14} \div 1\frac{13}{35}$, $1\frac{7}{18} \div 2\frac{8}{27}$, $4\frac{19}{27} \div 6\frac{5}{36}$.
9. $\dfrac{\frac{2}{35}}{\frac{7}{10}}$, $\dfrac{\frac{9}{14}}{6}$, $\dfrac{13}{\frac{2}{3}}$, $\dfrac{3\frac{1}{10}}{2\frac{7}{15}}$.
10. $\dfrac{\frac{2}{2\frac{1}{7}}}{\frac{2\frac{1}{7}}{2}}$, $\dfrac{\frac{1\frac{1}{3}}{4}}{\frac{3}{2\frac{1}{8}}}$, $\dfrac{\frac{3}{8}}{\frac{4\frac{1}{3}}{2\frac{1}{9}}}$, $\dfrac{\frac{2\frac{3}{8}}{6\frac{3}{4}}}{\frac{8\frac{1}{16}}{10\frac{1}{12}}}$.
11. $(3\frac{1}{3}-1\frac{1}{4}) \div 1\frac{5}{8}$, $7\frac{5}{6} \div \left(3\frac{3}{8}+\frac{5}{6}\right)$.
12. $(10\frac{1}{6}-2\frac{3}{8}) \div \left(\frac{3}{5} \times 6\frac{1}{3}-3\frac{3}{10}\right)$.
13. $\dfrac{12\frac{1}{2}-3\frac{5}{8}}{6\frac{7}{16}}$, $\dfrac{2\frac{3}{4}}{2\frac{1}{3}-\frac{5}{8}}$.
14. $\dfrac{12\frac{3}{7}-3\frac{5}{14}}{5\frac{1}{4}+2\frac{3}{7}}$, $\dfrac{12\frac{1}{3} \times 2\frac{5}{8}}{7\frac{3}{7}+\frac{13}{14}}$.
15. $(3\frac{1}{2}-2\frac{3}{8}) \times \dfrac{6\frac{2}{7}}{3\frac{1}{3}-1\frac{4}{15}}$.
16. $\dfrac{13\frac{1}{2}-8\frac{5}{8}}{2\frac{1}{10}-1\frac{7}{8}} \div \left(4\frac{1}{2} \times 3\frac{1}{3}-\frac{11}{12}\right)$.
17. $\dfrac{(3\frac{1}{3}+7\frac{5}{12}) \times \left(4-\frac{4}{7}\right)}{\left(1\frac{3}{8}-\frac{7}{8}\right) \div \left(1\frac{3}{8} \times \frac{7}{8}\right)}$.
18. $\dfrac{\left(2\frac{1}{7}-\frac{13}{35}-1\frac{1}{10}\right) \div \left(2\frac{1}{3}-1\frac{3}{4}\right)}{\left(\frac{13}{14} \div \frac{1}{5}\right) \div \left(3\frac{1}{3} \times \frac{3}{5} \div 1\frac{3}{4}\right)}$.

19. $\dfrac{1}{2+\dfrac{1}{2+\dfrac{1}{2+\dfrac{1}{2}}}}$, $\dfrac{2}{3+\dfrac{2}{5+\dfrac{2}{7+\dfrac{2}{9}}}}$. 20. $\dfrac{1}{1+\dfrac{1}{2+\dfrac{3^2}{2+\dfrac{5^2}{2}}}}$, $\dfrac{2\frac{1}{2}}{3\frac{1}{2}+\dfrac{2\frac{1}{2}}{3\frac{1}{2}+\dfrac{2\frac{1}{2}}{3\frac{1}{2}+\dfrac{2\frac{1}{2}}{3\frac{1}{2}}}}}$.

104. Approximate Simpler Forms of a Fraction.—It has already been remarked that when a fraction has in its simplest form a large numerator and denominator, and an idea of its magnitude is not easily obtainable, we may with advantage consider simpler forms which are *approximately* equal to it. How these *approximate simpler forms* may be got will now be shown.

Example 1. Find an approximate simpler form of the fraction $\frac{35}{52}$.

Although 35 has no factor, except unity, in common with 52, 36 has. Now $\frac{35}{52}$ is very nearly equal to $\frac{36}{52}$, and $\frac{36}{52} = \frac{9}{13}$, which is thus an approximate simpler form of $\frac{35}{52}$.

Trying for another number near 35 which has a factor in common with 52 we light upon 39. Thus—

$$\frac{35}{52} = \frac{39}{52} - \frac{4}{52}$$
$$= \frac{3}{4} - \frac{1}{13}.$$

$\frac{3}{4}$ is therefore another approximate simpler form of $\frac{35}{52}$, the difference between it and $\frac{35}{52}$ being only $\frac{1}{13}$.

Conversely we may seek for numbers nearly equal to the denominator and having a factor in common with the numerator. Thus—

$$\frac{35}{52} = \frac{35}{50} \text{ nearly,}$$
$$= \frac{7}{10} \text{ nearly;}$$
$$\text{and } \frac{35}{52} = \frac{35}{49} \text{ nearly,}$$
$$= \frac{5}{7} \text{ nearly.}$$

Further, we may seek for two numbers having a common

factor, one of which lies near the numerator and the other near the denominator. Thus—

$$\frac{35}{52}=\frac{34}{51} \text{ nearly,}$$
$$=\frac{2}{3} \text{ nearly.}$$

EXERCISES. SET LIII.

Find approximate simpler forms of the following fractions:—

1. $\frac{19}{24}$. 2. $\frac{92}{105}$. 3. $\frac{299}{700}$. 4. $\frac{49}{108}$. 5. $\frac{37}{65}$. 6. $\frac{115}{171}$.

105. There is another mode of finding approximate simpler forms of fractions which is in itself more important and more interesting than the above. To explain it, let us consider the fraction $\frac{16}{59}$.

Dividing numerator and denominator by the numerator we have—

$$\frac{16}{59}=\frac{1}{3+\frac{11}{16}}$$

from which if we reject the $\frac{11}{16}$ we have $\frac{1}{3}$ as an approximate simpler form of $\frac{16}{59}$.

Treating now the $\frac{11}{16}$ as we treated the given fraction we have—

$$\frac{16}{59}=\frac{1}{3+\frac{1}{1+\frac{5}{11}}}$$

and rejecting from this the $\frac{5}{11}$ we have another approximate simpler form, viz., $\frac{1}{3+\frac{1}{1}}$, or $\frac{1}{4}$.

Continuing the process, we find—

$$\frac{16}{59}=\frac{1}{3+\frac{1}{1+\frac{1}{2+\frac{1}{5}}}}$$

and rejecting from this the $\frac{1}{5}$ there results a third approxi-

mate simpler form, viz., $\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2}}}$, or $\frac{3}{11}$. The process evidently cannot be pursued farther.

The numbers 3, 1, 2, 5 occurring in the peculiar expression from which we get these approximate simpler forms are evidently the quotients obtained on dividing 59 by 16, this divisor by the remainder 11, this second divisor by the second remainder 5, and so on—exactly as if we were seeking the greatest common measure of 16 and 59. The work of finding them may thus be considerably shortened.

Example 2. Find the approximate simpler forms of $\frac{26}{67}$.

The quotients obtained in the ordinary process of finding the greatest common measure of 26 and 67 are 2, 1, 1, 2, 1, 3. Therefore

$$\frac{26}{67}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3}}}}}},$$

whence we have as approximations to $\frac{26}{67}$ the fractions

$$\frac{1}{2},\quad \cfrac{1}{2+\cfrac{1}{1}},\quad \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1}}},\ \&c.;$$

or, on simplification, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{5}$, &c.

A fraction having a complex form like several of those preceding is called a *continued fraction*, which may be defined as *a fraction whose denominator is a number increased or diminished by a fraction, this second fraction having its denominator a number also increased or diminished by a fraction, and so on to any length.* When the successive numerators are each 1, the fraction is called *a continued fraction with unit numerators.* This last is

evidently the kind of continued fraction into which we have changed the numbers above in order to find approximate simpler forms of them. The simpler forms found are called *convergents of the continued fraction*, or *convergents to the given number*, because each one can be shown to be a closer approximation than the one preceding it, and indeed closer than any fraction with a lower denominator.

EXERCISES. SET LIV.

Express the following as continued fractions with unit numerators :—

1. $\frac{17}{30}$, $\frac{30}{17}$. 2. $\frac{24}{37}$, $\frac{37}{14}$. 3. $\frac{34}{55}$, $\frac{45}{116}$.

4. $\frac{109}{142}$, $\frac{119}{256}$. 5. $\frac{157}{225}$, $\frac{1561}{1430}$. 6. $\frac{300}{199}$, $\frac{611}{743}$.

Find the convergents of the continued fractions—

7. $\dfrac{1}{4+\dfrac{1}{5+\dfrac{1}{6}}}$. 8. $2+\dfrac{1}{2+\dfrac{1}{4+\dfrac{1}{2+\dfrac{1}{4}}}}$. 9. $3+\dfrac{1}{7+\dfrac{1}{15+\dfrac{1}{1+\dfrac{1}{26}}}}$.

Find a series of convergents to the fractions—

10. $\frac{113}{235}$. 11. $\frac{445}{612}$. 12. $\frac{19}{30}$. 13. $\frac{169}{408}$.

EXAMINATION PAPERS ON §§ 64—109.

I.

1. 3 yd. 2 ft. of wire cost $2\frac{3}{4}$d. At this rate what length of wire might be bought for half-a-crown ?

2. Find the prime factors of 12, 20, 28, 32, 36, and thence obtain the lowest common multiple of these five numbers.

3. Find the greatest common measure of 112, 140, 189.

4. Perform the operations indicated in the expression—

$$(6\tfrac{1}{2}-3\tfrac{5}{8}+4\tfrac{3}{4})+(1\tfrac{1}{3}\times 2\tfrac{3}{8}).$$

5. What fraction is $4\frac{1}{2}$d. of $3\frac{1}{2}$ florins ?

6. Express 1 gr. in terms (1) of the *lb. avoird.* and (2) of the *oz. troy*.

7. Find approximate simpler forms of $\frac{18}{30}$.

8. The half of a sum of money is divided equally among five persons, each of whom thus receives an eighth of a shilling. Find the original sum.

II.

1. A piece of work may be done in 12 ho. 40 min. by 16 men, but 36 men are started upon it. What time will they take, supposing all the men mentioned work at the same rate?

2. Find the prime factors and the integral measures of 112.

3. Find the lowest common multiple of 361 and 380.

4. Perform the operations indicated in the expression

$$\left(\frac{2\frac{1}{2}}{3\frac{1}{2}}+\frac{4\frac{1}{3}}{4\frac{1}{4}}-\frac{1\frac{1}{2}}{2}\right) \div \left(4\frac{1}{2}\times 2\times\frac{8\frac{1}{3}}{1\frac{4}{5}}\right).$$

5. What fraction of $1\frac{1}{2}$ cwt. is 6 st. 12 lb.?

6. Express $\frac{9}{11}$ lb. in terms of the *cwt.*, and $\frac{9}{11}$ cwt. in terms of the *lb.*

7. Prove that $\frac{3}{8}\times\frac{2}{5}=\frac{3}{20}$.

8. After walking five-eighths of the distance between two towns, there remains to be walked a quarter of the distance and six miles more. What is the distance between the towns?

III.

1. If two men or three lads could dig a piece of ground in 12 days, what time would 1 man and 1 lad take to do it?

2. Find the lowest common multiple of the multiples of 7 which lie between 20 and 60.

3. Find the greatest common measure of the measures of 60 which are greater than 12.

4. Perform the operations which are indicated in the expression—

$$\left(\frac{3\frac{1}{2}-1\frac{1}{6}}{1\frac{3}{4}}+\frac{4\frac{1}{2}-2\frac{5}{6}}{2\frac{5}{6}}-\frac{2\frac{1}{4}+6\frac{1}{3}}{10\frac{1}{10}}\right)\times\frac{3}{4+\dfrac{3}{4+\dfrac{3}{4}}}.$$

5. Find the fraction which the eleventh part of a guinea is of the ninth part of a pound.

6. Express $\frac{80}{167}$ as a continued fraction with unit numerators, and thence derive a series of fractions approximate to $\frac{80}{167}$.

7. Prove that $\frac{3}{8}\div 1\frac{3}{4}=\frac{3}{14}$.

8. The first of three fractions is half of the second, the third is $\frac{5}{6}$, and the sum is $1\frac{23}{36}$. Find the first fraction.

DIFFERENT WAYS OF EXPRESSING THE SAME MAGNITUDE.

(*Continued from p.* 52.)

106. The knowledge of fractional numbers which we now possess enables us to obtain a more complete understanding of a subject already partially developed, viz., the various forms in which one and the same quantity may be expressed.

We know well already, for example, that "3 ft. 4 in." and "40 in." are two forms of expressing the same quantity, two units of measurement being used in the one case and a single unit in the other. Hitherto, however, if a number was given in terms of several units, and it was desirable to express it in terms of one, we could only, as in the example just given, employ for this purpose a unit not higher than the lowest of the several units mentioned; 3 ft. 4 in. we could express in terms of the *inch* alone, but not in terms of the *foot* alone, or the *yard* alone. This difficulty now no longer exists; 3 ft. 4 in. clearly may be expressed as $3\frac{1}{3}$ ft., and also, as we shall soon see, in the form $1\frac{1}{9}$ yd.

The object now before us, then, is to consider the various additional forms which fractions thus afford us of expressing any given quantity.

107. In the first place the quantity when given may be expressed in terms of a single unit.

Example 1. Express 41 oz. avoirdupois in terms (1) of the *pound*, (2) of the *cwt.*, and (3) of the *ton.*

(1) $$1 \text{ oz.} = \frac{1}{16} \text{ lb.}$$

$$\therefore \quad 41 \text{ oz.} = \frac{41}{16} \text{ lb. or } 2\tfrac{9}{16} \text{ lb.}$$

(2) The number of oz. in a cwt. is 16×112, or 1792, so that—

$$1 \text{ oz.} = \frac{1}{16 \times 112} \text{ cwt.}$$

$$\therefore \qquad 41 \text{ oz.} = \frac{41}{1792} \text{ cwt.}$$

(3) The number of oz. in a ton is $16 \times 112 \times 20$, or 35840, so that—

$$1 \text{ oz.} = \frac{1}{16 \times 112 \times 20} \text{ ton,}$$

$$\therefore \qquad 41 \text{ oz.} = \frac{41}{35840} \text{ ton.}$$

The primary requirement in such cases, viz., where a change is made from one unit to another, evidently is to know how the two units are related. If the change be from the shilling to the guinea, we must know that 1 shilling $= \frac{1}{21}$ of a guinea; if it be from the shilling to the farthing, we start with the fact that 1 shilling $= 12 \times 4$ or 48 farthings.

Example 2. Express $\frac{7}{18}$ yd. in terms (1) of the *foot*, (2) of the *inch*, (3) of the *mile*, and (4) of the usual variety of units.

(1) $$1 \text{ yd.} = 3 \text{ ft.}$$

$$\therefore \qquad \frac{7}{18} \text{ yd.} = \left(3 \times \frac{7}{18}\right) \text{ ft.} = 1\tfrac{1}{6} \text{ ft.}$$

(2) $$1 \text{ yd.} = 36 \text{ in.}$$

$$\therefore \qquad \frac{7}{18} \text{ yd.} = \left(36 \times \frac{7}{18}\right) \text{ in.} = 14 \text{ in.}$$

(3) $$1 \text{ yd.} = \frac{1}{1760} \text{ mi.}$$

$$\therefore \qquad \frac{7}{18} \text{ yd.} = \left(\frac{1}{1760} \times \frac{7}{18}\right) \text{ mi.} = \frac{7}{31680} \text{ mi.}$$

(4) $$\frac{7}{18} \text{ yd.} = 7 \text{ yd.} \div 18 = 1 \text{ ft. } 2 \text{ in.}$$

Exercises. Set LV.

Express—

1. £$\frac{3}{11}$, $\frac{3}{4}$d. in terms of the *shilling*.
2. $\frac{7}{9}$s., $\frac{4}{7}$d., $\frac{4}{9}$ guin. in terms of the £.
3. £$\frac{25}{96}$, $\frac{7}{16}$s. in the usual way of expressing a sum of money.
4. $\frac{3}{80}$ hr. in terms (1) of the *day*, (2) of the *second*, (3) of the *week*

5. $\frac{45}{56}$ cwt. in terms (1) of the *pound*, (2) of the *ton*.
6. $\frac{4}{11}$ lb. troy in terms of the *avoirdupois pound*.
7. $\frac{7}{11}$ fur. in terms (1) of the *yard*, (2) of the *mile*.
8. $\frac{25}{48}$ sq. po. in terms (1) of the *acre*, (2) of the *square inch*.
9. $\frac{9}{16}$ gall. in terms (1) of the *pint*, (2) of the *bushel*.
10. $2\frac{3}{7}$ oz. avoirdupois in terms of the *ounce troy*.

Express in the usual way, viz., by means of integral numbers and lower units—

11. $\frac{14}{27}$ da. 12. $\frac{43}{70}$ ton. 13. $\frac{7}{18}$ yd. 14. $\frac{16}{99}$ ro. 15. $\frac{7}{16}$ bus.
16. $2\frac{1}{16}$ mi. 17. $8\frac{1}{4}$ cub. yd. 18. $3\frac{8}{11}$ ac. 19. $6\frac{4}{11}$ lb. 20. $3\frac{1}{10}$ po.

108. In the second place the quantity, when given, may be expressed in terms of the usual variety of units.

Example 1. Express 4 sq. yd. 3 sq. ft. in terms (1) of the *square pole*, (2) of the *acre*.

(1)
$$\begin{aligned} 4 \text{ sq. yd. } 3 \text{ sq. ft.} &= 39 \text{ sq. ft.} \\ &= \frac{39}{9 \times 30\frac{1}{4}} \text{ sq. po.} \\ &= \frac{52}{363} \text{ sq. po.} \end{aligned}$$

(2)
$$\begin{aligned} 4 \text{ sq. yd. } 3 \text{ sq. ft.} &= 39 \text{ sq. ft.} \\ &= \frac{39}{9 \times 30\frac{1}{4} \times 40 \times 4} \text{ ac.} \\ &= \frac{13}{14520} \text{ ac.} \end{aligned}$$

We have thus only to express the given quantity in terms of the lowest unit mentioned, and then change from this unit to the one required, as in the examples of the preceding paragraph.

Example 2. Express 12 lb. 3 oz. $2\frac{4}{15}$ dr. in terms of the *ton*.

$$\begin{aligned} 12 \text{ lb. } 3 \text{ oz. } 2\tfrac{4}{15} \text{ dr.} &= 3122\tfrac{4}{15} \text{ dr.} \\ &= \frac{3122\frac{4}{15}}{16 \times 16 \times 112 \times 20} \text{ ton,} \\ &= \frac{46834}{573440 \times 15} \text{ ton,} \\ &= \frac{23417}{4300800} \text{ ton.} \end{aligned}$$

Example 3. Express £2 6s. 4d. in terms (1) of the *shilling*, (2) of the *pound*.

(1) £2 6s. = 46s.
4d. = $\frac{1}{3}$s.

∴ £2 6s. 4d. = $46\frac{1}{3}$s.

(2) 6s. 4d. = 76d.
= £$\frac{76}{240}$ or £$\frac{19}{60}$

∴ £2 6s. 4d. = £$2\frac{19}{60}$.

Exercises. Set LVI.

Express—

1. 3s. 4d., 14s. $5\frac{1}{2}$d., £7 12s. 6d., £2 0s. $0\frac{1}{2}$d., in terms of the *pound* (£) alone.
2. $3\frac{3}{4}$d., 16s. $8\frac{1}{2}$d., £6 12s. 5d., in terms of the *shilling* only.
3. 14 st. 8 lb., 3 cwt. 7 st. 7 lb., 3 tons 1 cwt. 1 qr., in terms of the *hundredweight* only.
4. 10 hr. 16 min., 10 hr. 17 min. $8\frac{4}{7}$ sec., as fractions of a *day*.
5. 2 yd. 2 ft., 6 po. 2 yd., as fractions of a *furlong*.
6. 5 yd. 2 ft. 2 in., in terms of the surveyor's *link*.
7. 1 ac. 3 ro. 20 sq. po., 1 ro. 2 sq. po. $15\frac{1}{3}$ sq. yd., in terms of the *acre* only.
8. 3 qt. $1\frac{1}{2}$ pt., 2 pk. 1 gall. 3 qt., in terms of the *gallon* only.
9. 1 lb. $11\frac{3}{7}$ oz. (avoirdupois), in terms of the *troy pound*.
10. 1 lb. $11\frac{3}{7}$ oz. (troy), in terms of the *avoirdupois pound*.

109. In the third place, two quantities of the same kind may be given, each expressed in terms of one or more units, and we may be required to compare them by expressing the one in terms of the other.

Example 1. What fraction is £5 10s. $2\frac{1}{2}$d. of £6 12s. 3d.?

£5 10s. $2\frac{1}{2}$d. = 5290 far.
and £6 12s. 3d. = 6348 far.
so that 1 far. = $\frac{1}{6348}$ of £6 12s. 3d.
and ∴ 5290 far. = $\frac{5290}{6348}$ of £6 12s. 3d.
that is, £5 10s. $2\frac{1}{2}$d. = $\frac{5}{6}$ of £6 12s. 3d.

Example 2. Express 90° in terms of the *radial* which is nearly 57° 17′ 45″.

$$\left.\begin{aligned} 90^\circ &= 324000'' \\ 57^\circ\ 17'\ 45'' &= 206265'' \end{aligned}\right\}$$

$$\therefore\ 90^\circ = \frac{324000}{206265}\text{ rad.}$$

$$= 1\ \frac{7849}{13751}\text{ rad.}$$

Exercises. Set LVII.

1. Express £7 12s. 6d., 1s. $5\frac{1}{2}$d., as fractions of £100.
2. What fraction of 10 guineas is £1 5s. 9d.?
3. What fraction is £2 13s. $5\frac{1}{4}$d. of £2 16s. 3d.?
4. Express 1 ac. 3 ro. 17 sq. po. as a fraction of 6 ac. 3 ro. 9 sq. po.
5. Express 2 cwt. 1 st. as a fraction of 7 cwt. $3\frac{1}{2}$ st., and 5 cwt. 2 lb. 8 oz. as a fraction of 6 cwt. 15 lb. 8 oz.
6. What fraction is 1 fur. 20 po. $2\frac{3}{4}$ yd. of 1 mi. 200 yd.?
7. What fraction of 3 lb. 11 oz. (avoirdupois) is 3 lb. 11 oz. (Troy)?

(*Subject continued on p.* 151.)

Exercises. Set LVIII.

Miscellaneous.

1. Three-fifths, one-ninth, and two-fifteenths of a piece of work have been performed. How much of it remains to be done?
2. How much greater than $\frac{10}{27}$ is the fraction got by increasing this numerator and denominator by 7?
3. What fraction added to the three fractions $\frac{1}{14}$, $\frac{1}{35}$, $\frac{1}{15}$ produces $\frac{8}{15}$?
4. Perform the operations indicated in the expression—

$$(2\tfrac{1}{3} + 1\tfrac{5}{6} - \tfrac{11}{12}) + (4\tfrac{1}{6} + \tfrac{13}{30}).$$

5. The Kohinoor diamond weighed 180 carats, but in the cutting it lost $\frac{37}{72}$ of its weight. What did it weigh after the operation?
6. $\frac{35}{64}$ of a certain number is $\frac{15}{28}$. Find the number.
7. When a certain number is divided by $3\frac{1}{7}$ the quotient is $3\frac{2}{11}$. Find the number.
8. Simplify the expression—

$$(2\tfrac{1}{3} + 1\tfrac{5}{6} - \tfrac{11}{12}) \div (4\tfrac{1}{6} \times \tfrac{13}{30}).$$

9. What is the cost of $\frac{5}{8}$ yd. of cloth at $\frac{2}{15}$s. per yard?
10. What would $9\frac{1}{8}$ cwt. of cotton cost at £$1\frac{13}{18}$ per cwt.?
11. $\frac{3}{7}$ ac. of ground cost £$1\frac{4}{15}$. At this rate, what would an acre cost?
12. What is the price of coal per ton when the price of $\frac{3}{14}$ cwt. is £$\frac{1}{35}$?

13. Simplify the expression—

$$\left\{5\tfrac{1}{2}-(2\tfrac{1}{3}-\tfrac{3}{4})\right\} \times (6\tfrac{3}{4}+5\tfrac{1}{18}).$$

14. A tourist in Austria bought $19\frac{1}{2}$ ells of cloth at $5\frac{5}{8}$ florins per ell. Reckoning the florin at $1\frac{23}{30}$s., find the cost of the cloth in English money.

15. A person who has lost $\frac{7}{18}$ of his fortune has £3960 still left. What sum has he lost?

16. A father left $\frac{5}{12}$ of his fortune to one son, $\frac{1}{8}$ to another, $\frac{1}{18}$ to the third, and the remainder, viz., £581 9s., to the fourth. What was the total sum left?

17. A stove consumes $25\frac{3}{8}$ lb. of coal per day. At this rate, how long will $6\frac{4}{7}$ cwt. of coal supply it?

18. Simplify the expression—

$$\frac{3}{1+\cfrac{1}{3+\cfrac{1}{3}}}+\frac{1}{4-\cfrac{3}{4-\cfrac{1}{2}}}.$$

19. A labourer did $\frac{3}{8}$ of a piece of work in $\frac{4}{7}$ hr. At this rate, what time would he spend on the whole work?

20. A person sells an estate for £81840, and thereby loses $\frac{3}{35}$ of what he paid for it. What did it cost him?

21. Three workmen are employed at a piece of work, of which one does $\frac{1}{10}$ per day, the second $\frac{1}{15}$, and the third $\frac{1}{30}$. How many days will they require to finish it?

22. $\frac{188}{255}$ of the surface of our globe is water, and Europe occupies $\frac{7}{100}$ of the land surface. What fraction of the whole surface does Europe occupy?

23. Find the difference between

$$\frac{2}{3-\cfrac{2}{3-\cfrac{2}{3-\cfrac{2}{3}}}} \quad\text{and}\quad \frac{3}{2+\cfrac{3}{2+\cfrac{3}{2+\cfrac{3}{2}}}}.$$

24. What fraction of the half of the half of the half of a thing is the tenth of the tenth of it?

25. Wine weighs $\frac{99}{100}$ of the weight of the same bulk of water, oil weighs $\frac{81}{88}$ of the weight of the same bulk of wine, and

"A pint of water
Weighs a pound and a quarter."

What is the weight of a pint of oil?

26. A person lost in one year $\frac{1}{4}$ of his fortune, and in the next $\frac{4}{7}$ of what remained, and then he was found to have £900. What did his original fortune amount to?

27. From a cask which is $\frac{3}{4}$ full, a flagon which holds $1\frac{7}{18}$ pt. is filled nine times, and then the cask is found to be only $\frac{3}{8}$ full. How many pints would the cask hold?

28. One man could do a piece of work in 16 hours, another could do it in 12 hours, and a third in 48 hours. In what time could it be accomplished by the three working together?

29. Simplify the expression—

$$\frac{\frac{1}{2}+\frac{2}{3}-\frac{5}{6}}{2+\frac{1}{2}-1\frac{1}{6}}+\left\{4\frac{1}{2}-\left(2\frac{1}{28}-1\frac{1}{3}\right)\right\}.$$

30. A jar contains $12\frac{3}{8}$ gall. of water, and $20\frac{5}{7}$ gall. of milk. How much water must there be in one gallon of the mixture?

31. A draper sells a piece of cloth at $\frac{17}{20}$ of the cost price, and thereby loses £$4\frac{1}{5}$. What did it cost him?

32. Simplify the expression—

$$\left\{(2\tfrac{1}{4})^2-(1\tfrac{1}{3})^3+(1\tfrac{1}{2})^4\right\}\div\left\{(2\tfrac{1}{4})^2\times(1\tfrac{1}{3})^3\times(1\tfrac{1}{2})^4\right\}.$$

33. A farmer buys two pieces of land at the same rate per acre; but one piece is $\frac{13}{20}$ of the other, and costs £1050 less. What did he pay for both?

34. A piece of work is performed by two workmen. The one having done only $\frac{5}{18}$ of it receives £$11\frac{1}{4}$ less than the other who has done the rest. How much does the latter receive?

35. Simplify the expression—

$$(4\tfrac{1}{10}-2\tfrac{3}{5})^2\times\frac{6\frac{1}{4}-\frac{16}{15}}{8\times(1\frac{1}{2}+3\frac{1}{4})}.$$

36. A vessel when full of water weighs $44\frac{3}{5}$ lb., and when $\frac{3}{7}$ of the water has been withdrawn it weighs only $35\frac{6}{7}$ lb. What does it weigh when empty?

37. Simplify the expression—

$$\frac{(7\frac{1}{4}-3\frac{1}{2})\times\left\{4\frac{1}{5}-(2\frac{1}{3}-1\frac{7}{10})\right\}}{(7\frac{1}{4}+3\frac{1}{2})\div(1\frac{1}{2}-9\frac{1}{2}\times\frac{9}{77})}.$$

38. A certain number is increased by $\frac{2}{3}$, the sum found is then multiplied by $\frac{2}{3}$, this product is next diminished by $\frac{2}{3}$, and the remainder thus obtained is divided by $\frac{2}{3}$; and $\frac{2}{3}$ is the quotient resulting. Find the original number.

39. An estate and £48300 were left to be divided equally between two persons, and, in accordance with this, one of them received $\frac{4}{7}$ of the money and $\frac{5}{13}$ of the land. What was the value of the estate?

40. Simplify the expression—

$$\frac{\left\{(2\frac{1}{10})^2\div(1\frac{3}{5}+\frac{9}{25})\right\}\times(1\frac{12}{17}-\frac{58}{51})^3}{\left[3\frac{1}{2}-\left\{3\frac{1}{2}-(3\frac{1}{2}-1\frac{2}{3})\right\}\right]\div\frac{4\frac{1}{5}}{3\frac{1}{2}+1\frac{1}{6}}}.$$

DECIMAL FRACTIONS.

EXPRESSION OF FRACTIONAL NUMBERS BY MEANS OF THE ORDINARY SYSTEM OF NOTATION FOR INTEGERS.

110. Having had hitherto to speak of fractions of *every* denomination,—*halves*, *thirds*, *quarters*, *fifths*, &c., it was utterly impossible to indicate the denominations in any other way than by explicitly writing them. This led to the introduction of a special notation for fractions, so that integral numbers were indicated according to one system, and fractional numbers according to another. Our object is now to show that by restricting ourselves to the consideration of fractions with denominators of a particular kind it is possible to indicate the denomination without explicitly writing it, and to embrace all numbers under one system of notation.

111. The fractions to which we thus limit our attention are DECIMAL fractions, that is, fractions whose denominators are powers of 10, *e.g.*, $\frac{8}{10}$, $\frac{7}{100}$, $\frac{8}{1000}$.

112. To see then how a simpler mode of writing these is suggested let us recur to the notation of integers. We know well that there the denominations *hundred*, *thousand*, &c., are not explicitly written but indicated merely by the *position* of the figures which tell the number of hundreds, thousands, &c., in the given integer. The fact, for example, of a figure being the farthest to the right of a group representing an integer is held to indicate that this figure specifies the number of separate or simple units in the integer. Now suppose for a moment that we make a slight change in this system, viz., indicate the place of simple units by a *special mark*, say an asterisk. As an example consider the number eighty-five, which would thus be written $8\dot{5}$ instead of 85. If immediately to the right of the figure 5 in the latter form we place the figure 7, thus,

$$8\dot{5}7$$

the effect is that the place of simple units is altered; instead of 8 tens and 5 simple units we have now 8 hundreds and 5 tens, and the 7 denotes 7 simple units. But by appending 7 in the same way to the form $8\dot{5}$ no such result follows. For in

$$8\dot{5}7$$

the 5 having the asterisk above it must still stand for 5 simple units and consequently the 8 for 8 tens; concerning the 7 we can say nothing, not having as yet contemplated the possibility of figures occurring to the right of the place of simple units. If, however, we now agree that units having such places shall be included in the convention already made, viz., that any unit is a tenth of the unit whose place is immediately to the left of that of the former, then the 7 has a perfectly definite value. For, just as the unit which occurs 5 times is a tenth of the unit which occurs 8 times, so the unit which occurs 7 times is a tenth of the unit which occurs 5 times; and the unit which occurs 5 times being the simple unit it follows that the unit which occurs 7 times is $\frac{1}{10}$. Thus $8\dot{5}7$ would denote $85+\frac{7}{10}$. Similarly the unit whose place is immediately to the right of the 7 is a tenth of $\frac{1}{10}$, that is, $\frac{1}{100}$; the next in order $\frac{1}{1000}$, and so on; the denominators being clearly the various powers of 10 in order; *e.g.*, $8\dot{5}701003$ would stand for $85+\frac{7}{10}+\frac{1}{1000}+\frac{3}{1000000}$. It is evident therefore that by adopting a special mark to indicate the place of simple units a possibility arises of expressing all decimal fractions after the manner of integers.

113. Instead of an asterisk the special mark used to indicate the place of simple units is in this country a round dot put immediately to the right of the place; for example,

$$4{\cdot}3=4\tfrac{3}{10},\quad .001=\tfrac{1}{1000}.$$

This dot separating the integral portion of the number from the fractional portion is called the ***units' mark*** or ***decimal point***.

114. Since, as we shall see, all decimal fractions can be expressed in this way, and all fractions of whatever denomination can be expressed to any degree of accuracy as decimal fractions, we have thus in the Arabic system of notation for integers a system applicable not only to integers but to numbers in general.

115. The learner is already perfectly familiar with this notation, the reading and writing of it, and the performing of the fundamental arithmetical operations with integral numbers written in it. The same practice remains now to be gone through with fractional numbers.

116. READING THE NOTATION.—We have already seen that numbers of several figures may be read in various ways, but that there is one way which is generally recognised.

The fundamental and simplest of all is that of naming the figures in order and adding to each the name of the unit whose place the figure occupies; for example, 84.3015 would in this way be read *eight tens four simple units three tenths one thousandth and five ten-thousandths.*

Secondly, we may separate the figures into groups in any way whatever, provided we do not alter the order, and then read each group as if it were a separate integer, adding in each case the name of the unit whose place is occupied by the last figure of the group; for example, separating the figures of the above number, thus—

84.|30|15

we read it as if written

"84 *units* 30 *hundredths* 15 *ten-thousandths;*"

or separating thus—

8|4.3|015

we read it as if written

"8 *tens* 43 *tenths* 15 *ten-thousandths;*"

or, again, thus—

84.301|5

and read it

"84301 *thousandths* 5 *ten-thousandths.*"

That each of the numbers now read is in reality 84.3015 is easily seen on examination. For example, the first of the three is

$$84+\frac{30}{100}+\frac{15}{10000}.$$

$$\left.\begin{aligned} &\text{Now } \frac{30}{100}=\frac{3}{10},\\ &\text{and } \frac{15}{10000}=\frac{10}{10000}+\frac{5}{10000}, \text{ or } \frac{1}{1000}+\frac{5}{10000}, \end{aligned}\right\}$$

$$\begin{aligned} \therefore\ 84+\frac{30}{100}+\frac{15}{10000}&=84+\frac{3}{10}+\frac{1}{1000}+\frac{5}{10000}.\\ &=84.3015. \end{aligned}$$

Of all these modes of reading, the one most generally adopted is that which follows from separating the figures to the right of the units' mark, as we do in the case of the same number of figures to the left; in other words, from assimilating the reading of fractions to the reading of integers. For example—

.6 is read "6 *tenths*."
.13 is read "13 *hundredths*."
.417 is read "417 *thousandths*."
.0153 is separated thus,

.015|3,

and read

"15 *thousandths* 3 *ten-thousandths*."

.50074685 is separated thus,

.500|746|85,

and read

"500 *thousandths* 746 *millionths* 85 *hundred-millionths*."

.000015300002 7 is separated thus,

.000015|300002|7,

and read

"15 *millionths* 300002 *billionths* 7 *ten-billionths*."

EXERCISES. SET LIX.

Read or write in words the numbers indicated as follows:—

1. .007, .03, .0004, .000001, .00005.
2. .0000009, .000000007, .000000000003.
3. 1.05, 10.1, 200.02, 3000.0003.
4. .31, .312, .3123, .03123, .003013.
5. .000312, .0013, .231231, .618186.
6. .130019, .300003, .012012012.
7. .0500607, .0004001, .10000005.
8. 10000000.0000007, .0000110001.
9. .000006304027, .007300001002.
10. .000010001001, .00000010000001.

117. WRITING THE NOTATION.—Here, as in the preceding exercises, the fundamental requirement is a knowledge of the places of the various units. As a step towards this, let it be remembered that the *third* place to the right of the units' mark is the place of *thousandths*, the *sixth* place that of *millionths*, the *ninth* that of *thousand-millionths*, the *twelfth* that of *billionths*, &c. These being known, no difficulty will be felt regarding the places of the intermediate units. The figures for the integral portion of the given number named are of course first written, and the units' mark placed at the end; then the figures denoting the number of thousandths, or whatever may be the first unit mentioned in the fractional part, are so placed that the last of them is in this unit's place; and so on with the other units mentioned, vacant places being filled up in the manner already known, viz., by zeros. For example, if the number be *three hundred and fourteen, twelve thousandths five millionths and three hundred-millionths*, we first write the integral portion and append the units' mark, thus

314.

then we place the figures 12, denoting the number of thousandths, so that the last figure, 2, is in the thousandths' place, and fill the vacant tenths' place with a zero, thus

314.012

and the millionths and hundred-millionths which are mentioned being similarly taken note of, we have finally

314.01200503.

Again, suppose the number were *ten thousandths two hundred millionths four thousand six hundred and two billionths.* There being no integral portion, we should first write the units' mark; then, the number of thousandths being 10, we should place the figures 10 so as to have the 0 in the third right-hand place, filling up as before the first right-hand place with a zero; next, the number of millionths being 200, we should write these three figures in order with the last of them in the sixth right-hand place; lastly, the number of billionths being 4602, the 2 would be written in the twelfth right-hand place, with the three other figures in order before it, and the two vacant places (viz., the seventh and eighth) would be filled with zeros. We should thus have

.010200004602.

Exercises. Set LX.

Express the following numbers in the general decimal notation—

1. Three *tenths*; four *thousandths*; nine *millionths*; seven *hundred-thousandths.*

2. One *hundred-millionth*; five *billionths*; seven *thousand-billionths.*

3. Four *whole units* and seven *tenths*; sixteen *whole units* and three *hundredths*; nine thousand *whole units* and nine *thousandths*; one hundred *whole units* and one *ten-millionth.*

4. Fifteen *thousandths*; twenty-seven *hundred-thousandths*; four hundred and two *millionths*; one thousand and seventeen *billionths.*

5. Two hundred and three *thousandths* fourteen *millionths*; fifty *thousandths* seven *hundred-millionths*; six hundred and six *millionths* two thousand and twelve *billionths.*

6. Ten *millionths*; three thousand and three *millionths*; four hundred and seven *ten-thousandths*; two thousand and sixteen *hundredths.*

7. Fifteen *thousandths* two hundred and two *millionths*; one hundred and ten *thousandths* twelve *millionths*; fifteen *millionths* two hundred and two *billionths.*

8. Two hundred and five *tenths*; seventeen thousand and eleven *millionths*; ten *whole units* and four million three hundred *billionths.*

9. Thirty *thousandths* four hundred and sixteen *ten-millionths*; seventy-six *hundredths* two *ten-thousandths* one *millionth*; five hundred and four *ten-thousandths* eighteen *ten-millionths*.

10. Forty-one *whole units* two hundred and sixty *ten-thousandths*; one thousand and forty-three *millionths*; three *tenths* two hundred and sixty-five *ten-millionths*.

118. It has been already remarked that decimal fractions may be found representing as approximately as we chose fractions of any other denomination whatever. Let us consider the fraction $\frac{3}{4}$. To represent it as a decimal fraction, the denominator must be changed from 4 to some power of 10; so that we have to find an integer such that, on multiplying the present denominator, 4, by it, the result will be 10, 100, 1000, or some higher power; in other words, we must find some power of 10 which is a multiple of 4. Now the lowest such power is 100, which contains 4 exactly 25 times, and we thus have

$$\frac{3}{4}=\frac{3\times 25}{4\times 25}=\frac{75}{100}.$$

Similarly,

$$\frac{15}{8}=\frac{15\times 125}{8\times 125}=\frac{1875}{1000}.$$

From this it is clear that the denominator of a fraction expressible as a decimal fraction must be contained an exact number of times in some power of 10; and as the powers of 10 can contain no prime factor except 2 and 5, it follows that, if the given denominator contains any other prime factor than 2 or 5, the fraction cannot be expressed as a decimal fraction. Thus, suppose the given fraction were $\frac{1}{6}$, where the prime factors of the denominator are 2 and 3; then the presence of the factor 3 enables us to say at once that there is no integer the product of which and 6 is a power of 10, and consequently that $\frac{1}{6}$ cannot be accurately expressed as a decimal fraction.

Exercises. Set LXI.

1. Find the lowest integer by which 2, 5, 25, 16, 125, 32, 3125, must be multiplied so as to produce a power of 10.

Express the following as decimal fractions in both notations—

2. $\frac{1}{2}, \frac{3}{5}, \frac{17}{25}$. 3. $\frac{3}{8}, \frac{17}{40}, \frac{11}{16}$. 4. $\frac{3}{32}, \frac{11}{125}, \frac{41}{1600}$. 5. $\frac{12}{625}, \frac{3}{31250}, \frac{63}{64}$.

6. What numbers under 20 are such that, when they are used as denominators, the fractions they belong to are not accurately expressible as decimal fractions?

Another way of treating this problem is as follows. $\frac{3}{4}$ is the fourth part of 3; and 3 being 30 tenths, the fourth part of 3 is 7 tenths, with 2 tenths still remaining to be divided; but 2 tenths is 20 hundredths, the fourth part of which is 5 hundredths. Thus, as the fourth part of 3 we find 7 tenths and 5 hundredths, or .75 as before.

Similarly, $\frac{15}{8} = 15 \div 8 = 1.875$

$\frac{1}{16} = 1 \div 16 = .0625.$

What this in fact amounts to is that we need not as hitherto stop our process of division at the place of simple units, since we now recognise units of a lower order, and have places for them farther to the right. For example, instead of saying that the result of dividing 275 by 8 is "34, and 3 remaining to be divided," we may now say that it is "34.3, and 6 tenths remaining to be divided," or "34.37, and 4 hundredths remaining to be divided," or that it is exactly 34.375.

EXERCISES. SET LXII.

Express the following as decimal fractions—

1. $\frac{13}{16}, \frac{117}{160}, \frac{124}{125}$. 2. $\frac{1}{25}, \frac{7}{80}, \frac{2}{125}$.

3. $\frac{19}{640}, \frac{3}{3200}, \frac{17}{3125}$. 4. $\frac{101}{12800}, \frac{1}{156250}, \frac{11}{102400}$.

Let us now examine in this way a fraction of the refractory kind referred to above (Ex. 6, Set LXI.). Although it be impossible to find a decimal fraction which shall be the exact equivalent of such a fraction, still we can always obtain one which is as close an approximation to it as may

be desired. Consider the fraction $\frac{3}{7}$. Dividing the 3 by the 7 we find

$$\frac{3}{7} = .428, \text{ with remainder 4 thousandths;}$$

that is,

$$\frac{3}{7} = .428 + \frac{4}{7000};$$

so that as a decimal fraction approximate to $\frac{3}{7}$ (viz., within less than a thousandth of it) we find

$$.428.$$

Proceeding, however, with the division farther than the third right-hand place, we have

$$\frac{3}{7} = .428571 + \frac{3}{7000000},$$

and thus for $\frac{3}{7}$ there is obtained a decimal fraction within less than a millionth of it, viz.,

$$.428571.$$

If we desire greater accuracy, we proceed and find .428571428571 with remainder 3 billionths; but we see, as indeed we know before starting, that we cannot hope ever to express by a finite number of figures the perfectly exact result. Such a decimal fraction, viz., one requiring for its expression an endless range of figures, is called *interminate*.

Observe, however, that there is no difficulty in finding the figures necessary to express the fraction to any degree of accuracy. For without continuing the division we can tell in order every figure of the series. The number of millionths remaining after the first six figures of the quotient were found being 3, *i.e.*, the very number we started with as dividend, it follows that the second six figures must be the same as the first six, so also of necessity the third six, and so on. A similar result occurs in the case of every number given in common fractional form if it require an interminate decimal fraction to express it. For in the process of division every remainder being necessarily less than the

divisor, some number must occur *twice* as a remainder before the number of figures found in the quotient is equal to the divisor. For example, in the case of the fraction $\frac{5}{11}$, each of the remainders in the process of division must be less than 11, that is to say, must be one of the first ten integers: consequently, when we have had ten remainders all the different remainders possible must have occurred, and the eleventh must be the same as one of those preceding it: thus a recurrence is inevitable before more than 11 figures of the quotient have been obtained. As a matter of fact, the recurrence takes place at the *third* figure, for

$$\frac{5}{11} = .454545\ldots\ldots$$

A decimal fraction of this kind, viz., one which is interminate on account of a group of figures in it being repeated without end or interruption, is called *periodic*. The group of figures which recurs is called the *period*, and in this country it is usual to indicate it by placing a dot over its first and last figures: for example, the above results are written

$$.\dot{4}2857\dot{1} \text{ and } .\dot{4}\dot{5}.$$

It is to be particularly noted, however, that in saying that $.\dot{4}2857\dot{1}$ is the equivalent of $\frac{3}{7}$, we can only mean that by continuing the proper figures farther and farther to the right of the units' mark, we thereby find a decimal fraction approximating more and more nearly to $\frac{3}{7}$, but that it can never $= \frac{3}{7}$ so long as the number of figures taken is finite. This, in mathematical language, is expressed by saying that $\frac{3}{7}$ is the *limit* of $.\dot{4}2857\dot{1}$.

Example 2. Find a decimal fraction whose limit is $\frac{9}{550}$.

$$\frac{9}{550} = 9 \div 550 = 0.01\dot{6}\dot{3}.$$

In the division we first ask how often 550 is contained in 9 (units), and put 0 in the simple units' place of the quotient as our answer; then, how often 550 is contained in

90 (tenths), putting 0 in the tenths' place of the quotient; then how often 550 is contained in 900 (hundredths), whence the 1 in the quotient; and so on. The remainder, 350, occurring for the second time, shows that the period is 63.

EXERCISES. SET LXIII.

Find decimal fractions within less than a thousandth of

1. $\frac{3}{7}, \frac{4}{11}, \frac{4}{9}$. 2. $\frac{9}{13}, \frac{4}{27}, \frac{18}{29}$.

Find decimal fractions within less than a millionth of

3. $\frac{4}{115}, \frac{2}{101}, \frac{11}{102}$. 4. $\frac{100}{103}, \frac{4}{1001}, \frac{16}{30011}$.

Find the decimal fractions whose limits are—

5. $\frac{2}{9}, \frac{5}{9}, \frac{8}{9}$. 6. $\frac{2}{3}, \frac{7}{15}, \frac{5}{12}, \frac{17}{45}$.

7. $\frac{31}{99}, \frac{14}{99}, \frac{2}{99}, \frac{3}{11}$. 8. $\frac{206}{999}, \frac{34}{999}, \frac{5}{999}$.

9. $\frac{16}{9999}, \frac{2014}{999999}$. 10. $\frac{17}{37}, \frac{5}{101}, \frac{141}{3367}$.

11. Find a decimal fraction within less than a millionth of the reciprocal of 11; of 13; of 17; and of 19.

12. Compare the numbers $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$ by approximately expressing them in the general decimal notation.

13. Write $.06\dot{6}$, $.14\dot{2}\dot{4}$, $.31842\dot{7}4\dot{2}$ in simpler forms as decimal fractions.

119. The converse operation of that which we have just explained, viz., expressing in the common fractional notation a fraction given in the general decimal notation, presents no difficulty if the number of figures in the given form be finite. Thus—

$$.0032 = \frac{32}{10000} = \frac{2}{625},$$
$$3.000015 = 3\frac{15}{1000000} = 3\frac{3}{200000};$$

the number of zeros in the denominators, as first written, being necessarily always the same as the number of figures to the right of the units' mark.

Exercises. Set LXIV.

Express the following fractions in the common fractional notation:—

1. .7, .03, .0005, .000008.
2. .75, .115, .0235, .002075.
3. .005, .0025, .00064, 2.5.
4. 2.25, 7.125, 3.375, 1.625.
5. 1.012, 14.015, 101.1875, 400.038125.

When the given decimal fraction is *interminate*, this mode of proceeding will give us as many *approximate* forms as we please, but for a perfectly accurate result would require an infinite number of figures in both numerator and denominator. Thus, for $.2\dot{1}\dot{7}$ we have the approximate forms $\frac{217}{1000}$, $\frac{2171}{10000}$, $\frac{21717}{100000}$, $\frac{217171}{1000000}$, &c. We shall now see, however, that for any interminate decimal fraction which is periodic, a fraction can be found with a finite number of figures in numerator and denominator which shall be the *equivalent*, in the sense already explained, of the given number. One series of such decimal fractions and their *limits* we are already familiar with, viz.,

$$.\dot{1}=\tfrac{1}{9},\quad .\dot{0}\dot{1}=\tfrac{1}{99},\quad .\dot{0}0\dot{1}=\tfrac{1}{999},\ \&c.;$$

and, knowing these, there is little difficulty with any others that may be given.

Consider first those in which the first figure of the period is in the tenths' place, *e.g.*, $.\dot{2}5\dot{3}$, $.\dot{0}01\dot{6}$.

$$\begin{aligned}
.\dot{2}5\dot{3} &= .253253253\ldots \\
&= .001001001\ldots \times 253 \\
&= \frac{1}{999}\times 253 \\
&= \frac{253}{999}.
\end{aligned}$$

$$\begin{aligned}
.\dot{0}01\dot{6} &= .\dot{0}00\dot{1}\times 16 \\
&= \frac{1}{9999}\times 16 \\
&= \frac{16}{9999}.
\end{aligned}$$

Clearly in such cases the fraction sought has for the figures of its denominator as many 9's as there are recurring figures, and for its numerator the integer formed by the recurring figures themselves. Thus we have at once

$$5.\dot{0}7\dot{1}=5\tfrac{71}{999}$$
$$12.\dot{2}081\dot{4}=12\tfrac{20814}{99999}.$$

EXERCISES. SET LXV.

Find the numbers in the common fractional notation which give rise to the following forms:—

1. $.\dot{0}00\dot{1}$, $.\dot{0}00000\dot{1}$, $.\dot{0}0000\dot{1}$.
2. $.\dot{7}$, $.\dot{3}$, $.\dot{0}\dot{7}$, $.\dot{6}\dot{3}$, $.\dot{9}$, $.\dot{0}\dot{9}$.
3. $.\dot{6}1\dot{2}$, $.\dot{0}2\dot{1}$, $.\dot{0}0\dot{3}$, $.\dot{9}90\dot{9}$.
4. $4.\dot{0}3\dot{7}$, $.\dot{0}12\dot{1}$, $.\dot{0}212\dot{1}$, $.\dot{9}00900\dot{9}$.

Next let us take fractions in which the first figure of the period is not in the tenths' place, *e.g.*, $.8\dot{2}\dot{9}$, $.03\dot{4}1\dot{5}$—

$$.8\dot{2}\dot{9}=\frac{8.\dot{2}\dot{9}}{10}=8\tfrac{29}{99}\div 10$$
$$=\frac{821}{99}\times\frac{1}{10}$$
$$=\frac{821}{990}.$$
$$.03\dot{4}1\dot{5}=\frac{3.\dot{4}1\dot{5}}{100}=3\tfrac{415}{999}\div 100$$
$$=\frac{3412}{999}\times\frac{1}{100}$$
$$=\frac{3412}{99900}.$$

Here we multiply by such a power of 10 as will place the units' mark immediately in front of the first recurring figure, and then counterbalance this by indicating that the result is to be *divided* by the same power of 10; after which the process is dependent upon what precedes. It will be seen that the numerator as first found is the integer formed by the non-recurring and the recurring figures diminished by the integer formed by the non-recurring figures; and that the figures of the denominator are first a 9 for every recurring figure, and then a zero for every non-recurring figure.

That this must always be the case will be seen by conducting in the following way the simplification of the expressions $8\frac{29}{99} \div 10$, and $3\frac{415}{999} \div 100$, found above—

$$8\tfrac{29}{99} \div 10 = \frac{8 \times 99 + 29}{99} \div 10 \qquad 3\tfrac{415}{999} \div 100 = \frac{3 \times 999 + 415}{999} \div 100$$
$$= \frac{8 \times 100 - 8 + 29}{990} \qquad\qquad = \frac{3 \times 1000 - 3 + 415}{99900}$$
$$= \frac{829 - 8}{990}. \qquad\qquad = \frac{3415 - 3}{99900}.$$

Other examples:—

$$.16\dot{8} = \frac{168 - 16}{900} = \frac{152}{900} = \frac{38}{225}.$$
$$13.0010\dot{0}\dot{1} = 13\frac{1001 - 10}{990000} = 13\tfrac{991}{990000}.$$

EXERCISES. SET LXVI.

Find the numbers in the common fractional notation which give rise to the following forms—

1. $.8\dot{3}$, $.8\dot{1}\dot{6}$, $.4\dot{7}\dot{2}$, $.31\dot{4}8\dot{6}$. 2. $.03\dot{1}\dot{4}$, $.002\dot{6}0\dot{3}$, $.00\dot{4}\dot{5}$.
3. $2.0\dot{0}0\dot{6}$, $2.00\dot{0}\dot{6}$, $2.000\dot{6}$. 4. $.5\dot{0}\dot{5}$, $.7\dot{0}0\dot{8}$, $.078\dot{0}\dot{8}$.

Find the limit of the following interminate expressions—

5. $\frac{8}{10} + \frac{8}{10^2} + \frac{8}{10^3} + \dots$ 6. $\frac{7}{10^2} + \frac{7}{10^4} + \frac{7}{10^6} + \dots$
7. $\dot{9}$. 8. $.00\dot{9}$. 9. $.002\dot{9}$. 10. $5.374\dot{9}$.

120. The operations of Addition, Subtraction, Multiplication, and Division, which the learner can already perform with integral numbers, may now, in a perfectly similar way, be performed with any numbers whatever expressed in the general decimal notation. If, however, among the given numbers there occur a decimal fraction distinctly marked periodic, which is of extremely rare occurrence in practical affairs, there is usually much saving of time and labour in using, in place of it, its equivalent in the ordinary fractional notation. This especially holds in cases where, in the result, nothing but accuracy to the utmost limit would be considered sufficient, a requirement, it need scarcely be added, of equally rare occurrence. Further, as is clear from this, if the numbers be given in the ordinary fractional notation,

it would in general be a still greater increase of labour to change them into the general decimal notation before performing the desired operations.

121. ADDITION.—Example 1. Find the sum of 6.3104, 218.01, .754168, .090134, 8001.62, and 75.8.

$$\begin{array}{r@{.}l} 6 & 3104 \\ 218 & 01 \\ & 754168 \\ & 090134 \\ 8001 & 62 \\ 75 & 8 \\ \hline \text{Sum} = 8302 & 584702 \end{array}$$

Here care is only necessary that, for convenience in the addition, the tenths' place of one item be exactly below the tenths' place of the preceding item, the hundredths' place below the hundredths' place, and so on.

Example 2. What is the sum of $16.5\dot{1}\dot{6}$, $.\dot{8}2\dot{3}$, and $316.66\dot{7}$?

In our mode of answering a question of this kind we must be guided by the degree of accuracy the questioner may require. If, for example, a result correct in the number of millionths be deemed sufficient, we should use the items—

$$\begin{array}{r@{.}l} 16 & 5161616 \\ & 8238238 \\ 316 & 6677777 \end{array}$$

each of which is less than the corresponding actual item by less than a ten-millionth. The sum thence found is

$$334.0077631$$

which is consequently less than the actual sum by less than 3 ten-millionths. Now the addition of 3 ten-millionths to it would not alter the figure in the millionths' place; therefore 334.007763 is correct in the number of millionths, or "correct to six right-hand places." Speaking of such cases as are likely to occur, we may say that if the sum

be desired correct to a given number of places it is advisable to use items correct to two places more.

If, on the other hand, a perfectly accurate result be required, we continue to write the recurring figures of the items until a recurrence is observed among the columns; thus—

$$\begin{array}{r}
16.5161616161 6\ldots \\
.8238238238 2\ldots \\
316.6677777777 7\ldots \\
\hline
334.0077632177 5\ldots
\end{array}$$

Here we see that the six columns lying between the two upright lines recur in the same order without end or interruption, and thus it is clear that the exact sum must be

$$334.00\dot{7}7632\dot{1}.$$

EXERCISES. SET LXVII.

Perform the following additions—

1. $173.135 + 2.0046 + 6843.2 + .000163 + 24.002.$
2. $38.261 + 3.80004 + 2100.613 + 2.32696.$
3. $7.3264 + 2.32925 + .0065 + .00085.$
4. $.213256 + .103425 + .002847 + .680472.$

5. Add the following numbers:—Seven *whole units* and three *tenths*, one *whole unit* and seventeen *hundredths*, thirteen *whole units* and one hundred and twelve *thousandths*, sixteen *thousandths* three hundred and eleven *millionths*, three hundred *thousandths* and nineteen *millionths*.

6. Add the following numbers:—Twenty-five *thousandths* seventeen *millionths*, three hundred and four *thousandths* thirty-three *hundred-thousandths*, seven *whole units* and sixteen *hundredths*, two *whole units* and one hundred and fifty-three *millionths*.

7. Add the following numbers:—Five *ten-thousandths*, twelve *thousandths* two *millionths*, nineteen *ten-millionths*, three hundred and sixty-two *hundred-thousandths*.

8. Add the following numbers:—One *thousandth* seventy-five *thousand-millionths*, six hundred *millionths* seven *hundred-millionths*, one hundred *thousandths* two hundred *millionths* four thousand and one *billionths*.

9. Find the sum of $.\dot{6}$, $.31\dot{4}\dot{2}$, 6.34, $2.0\dot{8}0\dot{8}$, correct to five right-hand places.

10. Find the sum of $.\dot{2}3\dot{4}$, $.0\dot{9}$, 3.15, $3.\dot{6}\dot{9}$, $.116\dot{4}\dot{3}$ correct to within less than a millionth.

11. Find the sum of $.0\dot{2}\dot{9}$, $.5\dot{6}$, $.954\dot{8}$, $.\dot{9}\dot{4}$ correct to within less than a billionth.

12. Find the exact sum of $2.\dot{0}\dot{1}$, $.0\dot{3}0\dot{3}$, $.0\dot{3}$, and $4.161\dot{4}$.

13. Find the exact sum of $.302\dot{6}$, $.7\dot{4}\dot{2}$, $.\dot{8}2\dot{3}$, and $.4\dot{0}62\dot{6}$.

14. Find the exact sum of $.5\dot{0}\dot{6}$, $.\dot{3}\dot{5}$, $.39\dot{4}$, $.7468\dot{9}$.

15. Find the exact sum of $3.4\dot{9}\dot{4}$, $.74\dot{6}\dot{4}$, $13.304\dot{9}$, $999.\dot{9}$.

122. SUBTRACTION.—Example 1. From 200.713 take 18.304.

$$\left.\begin{array}{r} 200.713 \\ -\ 18.304 \end{array}\right\} = 182.409.$$

Example 2. What is the difference between .734168 and 12.2?

$$\left.\begin{array}{r} 12.2 \\ -\ .734168 \end{array}\right\} = 11.465832.$$

Example 3. Find the difference between $2.\dot{6}1\dot{3}$ and $.8\dot{4}\dot{7}$ correct (1) to five right-hand places, (2) perfectly.

(1)
$$\left.\begin{array}{r} 2.6136136\ldots \\ -\ .8474747\ldots \end{array}\right\} = 1.76613\ldots$$

(2)
$$\left.\begin{array}{r} 2.61361361 3\ldots \\ -\ .847474747\ldots \end{array}\right\} = 1.7\dot{6}6138\dot{8}.$$

EXERCISES. SET LXVIII.

Perform the operations indicated as follows:—

1. 34.6173 − 17.2985.
2. .026814 − .003965.
3. .00174 − .00038.
4. .001762 − .000949.
5. 1.0001 − .90098.
6. .100005 − .099999.
7. 3184.7 − 216.35.
8. 2.164 − .92931.
9. 12 − .3406127.
10. 100 − .0000001.
11. (.0137 − .0000849) − (1.5003 − 1.4999901).

12. From thirty *thousandths* fourteen *millionths* take away three hundred and seventy *millionths* four thousand two hundred and one *billionths*.

13. Find the difference between $.0\dot{0}\dot{9}$ and $2.\dot{6}$, and between $1.\dot{6}9\dot{2}$ and $2.3\dot{0}\dot{4}$ correct to six right-hand places.

14. Find the exact difference between $2.\dot{1}\dot{6}$ and $3.04\dot{2}81\dot{3}$, and between $3.08\dot{2}\dot{1}$ and $4.3\dot{0}\dot{9}$.

15. Find the exact difference between $1.54\dot{9}4\dot{2}$ and $.\dot{9}\dot{4}$, and between $6.3\dot{2}\dot{7}$ and $4.\dot{7}2\dot{6}$.

123. MULTIPLICATION.—Example 1. Find the product of 232.341 and 43.62.

$$\begin{array}{r}
232.341 \\
\times \quad 43.62 \\
\hline
4.64682 \\
139.4046 \\
697.023 \\
9293.64 \\
\hline
10134.71442
\end{array}$$

Here we first multiply by 2 hundredths. Now thousandths multiplied by hundredths result in hundred-thousandths, therefore the 2 arising from the multiplication of the 1 of the multiplicand by the 2 of the multiplier is made to occupy the fifth place to the right of the units' mark; and similarly the position of any other figure is explained —exactly as in the case of integral numbers.

The figuring in the above is quite the same as if the multiplicand and multiplier had been 232341, 4362 instead of 232.341, 43.62. The units' mark may thus be entirely overlooked until all the figuring has been performed. Its position in the product may then be settled as above; or, without reasoning, by merely remembering that it should have as many figures on its right as there are figures so situated in both multiplicand and multiplier. That we may always proceed in this way will be seen from the following mode of viewing the multiplication.

$$\begin{aligned}
232.341 \times 43.62 &= \frac{232341}{1000} \times \frac{4362}{100} \\
&= \frac{232341 \times 4362}{100000} \\
&= \frac{1013471442}{100000} \\
&= 10134.71442.
\end{aligned}$$

Example 2. Multiply .00612 by 324000.

$$
\begin{array}{r}
.00612 \\
\underline{324000} \\
2448000 \\
1224 \\
\underline{1836} \\
198288000
\end{array}
$$

Now, there being five figures to the right of the units' mark in the multiplicand and none so situated in the multiplier the product must be 1982.88000 or 1982.88.

As a result of our notation being decimal, multiplication by a power of 10 is extremely easy, amounting merely to shifting the units' mark in the multiplicand one place to the right for every zero in the multiplier : *e.g.*,

$$2.34 \times 10 = 23.4$$
$$.00003 \times 1000 = \ .03$$
$$.000001 \times 10000000 = 10, \quad \text{\&c.};$$

just as in the case of integers

$$26 \times 100 = 2600$$
$$30 \times 1000 = 30000, \text{ \&c.}$$

Exercises. Set LXIX.

Perform the operations indicated as follows :—

1. $.07 \times 10$; $.07 \times 100$; $.07 \times 1000000$; 70.07×1000.
2. $.000101 \times 1000$; $.000101 \times 100000000$; $.0001 \times 10000$.
3. 714.2×21.6.
4. 1.3416×2.032.
5. $125000 \times .0008$.
6. $.0025 \times 36000$.
7. $.016 \times 187500$.
8. $.304 \times .00355$.
9. $.001 \times .0004$.
10. $.00302 \times .040004$.
11. $1.01 \times 1001 \times .003$.
12. $.00102 \times .002 \times 50000$.
13. $(30.2)^2 - (.08)^2$.
14. $(30.2 - .08) \times (30.2 + .08)$.
15. $(.004)^3 \times (7500)^3$.
16. $(.004 \times 7500)^3$.

123A. If a product be wanted correct to a certain number of right-hand places, fewer than would be required for perfect accuracy, the unnecessary figuring cannot be so easily foreseen and set aside as in the case of addition or subtraction. Let us consider the following example.

Example 1. Find the product of .18536472 and 2.1365489.

To insure accuracy to four right-hand places in the final result it is advisable to retain *six* right-hand places in the products which are found on multiplying by the various parts of the multiplier and which are afterwards to be added. Consequently the only point to be settled is, how much, if any, of the multiplicand may be neglected in multiplying by the parts referred to so as still to have six right-hand places in the product. Beginning with the 2 of the multiplier it is easily seen that with six right-hand places in the multiplicand there will be six right-hand places in the product, and we commence therefore to multiply at the figure 4 in the multiplicand, obtaining thus the product

.370728.

Passing on to the next part of the multiplier, viz., .1, we see that with five right-hand places in the multiplicand there will be six right-hand places in the product, and consequently we begin to multiply by 1 at the figure 6 in the multiplicand, the product obtained being

.018536.

Similarly we begin to multiply by 3 at the figure 3 of the multiplicand, by 6 at the figure 5 of the multiplicand, and so on. Making omissions, for brevity's sake, of units' marks and zeros, the necessary figuring may be arranged thus :—

$$
\begin{array}{r}
.18536472 \\
\underline{2.1365489} \\
.370728 \\
18536 \\
5559 \\
1110 \\
90 \\
\underline{4} \\
.3960\,..
\end{array}
$$

If we place temporarily the 2 of the multiplier over the sixth right-hand place of the multiplicand and the other figures of the multiplier so that the whole shall appear reversed in order, thus,

$$\begin{array}{r} {\scriptstyle 98456312} \\ .18536472 \\ \underline{2.1365489} \end{array}$$

it is clear that each figure of this reversed multiplier will be exactly over the place at which we begin to multiply by it, and to some this artifice may be of use in saving a little time and thought.

Example 2. Find the product of $.\dot{2}\dot{7}$ and $34.0\dot{1}08\dot{9}$ correct (1) to three right-hand places, (2) to the utmost.

(1)

$$\begin{array}{r} {\scriptstyle ...19801043} \\ .272727... \\ \underline{34.010891...} \\ 8.18181 \\ 1.09088 \\ 272 \\ \underline{16} \\ 9.275\,...... \end{array}$$

(2)

$$.\dot{2}\dot{7}.\times 34.0\dot{1}08\dot{9}=\frac{27}{99}\times\frac{3400749}{99990}$$
$$=\frac{3}{11}\times\frac{34351}{1010}$$
$$=\frac{103053}{11110}$$
$$=9.2\dot{7}56\dot{9}.$$

Exercises. Set LXX.

1. Find the product of .316278 and 7.21345, and the product of 2.741302 and 1.6428145 correct to four right-hand places.

2. Find the product of .2100634 and .641271, and the product of 3.1516283 and .310046 correct to three right-hand places.

3. Find the product of .061428134 and 114.204206, and the product of 31.260480707 and 204.010203069 correct to six right-hand places.

Find the products in the following cases correct (1) to three right-hand places, (2) to the utmost:—

4. $14.89\dot{3} \times .18\dot{6}$. 5. $.07\dot{3} \times .0\dot{1}\dot{8}$.
6. $.23\dot{1}\dot{8} \times 20.\dot{8}1\dot{0}$. 7. $.01\dot{5}\dot{7} \times 3.\dot{5}3846\dot{1}$.

124. DIVISION.—In this as in the other operations we may guide ourselves merely by the same general principles which are used in the case of integral numbers, but for the learner there is less chance of error in using an indirect method which takes advantage of the principle that the quotient is unaltered if divisor and dividend be both multiplied by the same number.

Example 1. Divide .03 by .007.

$$\begin{aligned} .03 \div .007 &= 30 \div 7 \\ &= 4.\dot{2}8571\dot{4}. \end{aligned}$$

Here we multiply dividend and divisor by such a power of 10 as will make both integral, viz., by 1000; in other words, we remove the units' mark three places to the right in both. There thus remains but to divide 30 by 7.

Example 2. Divide 4.263435 by .063.

$$\begin{aligned} 4.263435 \div .063 &= 4263435 \div 63000 \\ &= 67.673\dot{5}71428\dot{8}. \end{aligned}$$

Or we may aim only at having the *divisor* integral, thus

$$4.263435 \div .063 = 4263.435 \div 63$$

performing the division as follows:—

$$\begin{array}{r} 7\underline{)4263.435} \\ 9\underline{)609.062\dot{1}4285\dot{7}} \\ 67.673\dot{5}71428\dot{8} \end{array}$$

In many cases, however, this shifting of the units' mark is of little moment. It only serves to simplify the fixing of the place of the units' mark in the quotient, and this is often quite apparent on looking at the given numbers themselves. For example, if asked to divide 207.861 by 5.79, we see clearly on looking at the integral parts of the numbers that

there must be two integral places in the quotient, and we proceed directly as follows :—

$$
\begin{array}{r}
5.79)207.861(35.9 \\
\underline{173\ 7} \\
34\ 16 \\
\underline{28\ 95} \\
5\ 211 \\
\underline{5\ 211}
\end{array}
$$

Our notation being decimal, division by a power of 10 amounts merely to shifting the units' mark in the dividend as many places to the left as there are zeros in the divisor, *e.g.*,

$$
\begin{array}{ll}
38.723+100 & =.38723 \\
2.4\quad +10000 & =.00024 \\
16000+1000 & =16 \\
16000+1000000 & =.016.
\end{array}
$$

Exercises. Set LXXI.

Find the quotients in the following cases—

1. .01+10; .01+1000; .01+1000000.
2. 1010+100; 1010+10000; 1010+1000000.
3. 2021.2+1000; 2021.2+1000000000.

4. 172+430.
5. 377+29000.
6. 314.88+400.
7. .0082812+6.
8. .08961+103.
9. .00010101+3367.
10. .25+.64.
11. .023+.125.
12. 25+.125.
13. .23+.0125.
14. 764.2292+.191.
15. .00007+.00000016.
16. .0001+.008.
17. .01495+11.5.
18. .000021+.014.
19. .0003562+13.7.
20. .00002346+.051.
21. .0000002247+.01177.
22. $3.00\dot{2}+1.\dot{5}\dot{1}$.
23. $.2510\dot{1}\dot{6}+14.7\dot{6}\dot{5}$.
24. $1.\dot{5}7142\dot{8}+.\dot{2}\dot{7}$.
25. $.\dot{8}7\dot{3}+1.\dot{5}4\dot{9}$.

Examination Papers on §§ 110—124.

I.

1. From seven and seven millionths take away three tenths and nine ten-millionths.

2. Perform in both notations the multiplication of four thousandths by fifteen ten-millionths, and show that the results agree.

3. Express .000375 in its simplest form in the common fractional notation, and find a decimal fraction equal to $\frac{3}{32}$.

4. Perform in both notations the division of the fourth power of six tenths by the third power of four hundredths.

5. Perform the operations indicated in the expression—

$$(3.05 - 1.995) \times (.0005 + .025).$$

6. Multiply the difference between .075 and 3.04 by .0047619.

7. Find the product of .987654321 by .123456789 correct to four right-hand places.

II.

1. Find the excess of six and three thousand-millionths over two and one hundred and ninety-nine billionths.

2. Express in both fractional notations the excess of .035 over $\frac{11}{320}$.

3. Multiply the difference between the second power of five hundredths and the second power of five thousandths by four millionths.

4. Perform in both notations the division of ten and eleven thousandths by the excess of the second power of one tenth over the second power of two hundredths.

5. Prove that .153 is within less than a thousandth of $\frac{2}{13}$.

6. Perform the operations indicated in the expression—

$$\frac{4.3 - 2.005}{.001 - .0001} \times (.54 + .0027).$$

7. Find the sum of the infinite series—

$$.05 + (.05)^2 + (.05)^3 + (.05)^4 + \ldots,$$

correct to within less than a billionth.

III.

1. Perform in the general decimal notation the division of a thousandth of a thousandth of a thousandth by a millionth of a millionth.

2. Arrange the following numbers in order of magnitude:—

$$\frac{2\frac{1}{2}}{2\frac{1}{12}}, \quad \frac{48 + 1000}{.04}, \quad .0\dot{6}\dot{4} + .06\dot{4}, \quad .5\dot{6} + .\dot{5}\dot{6}.$$

3. If a dose of a homœopathic medicine be .025 scruple, how many doses are there in 2 oz. 5 dr. 1 scr. 4.5 gr. of the medicine?

4. Perform the operations indicated in the expression—

$$\left\{ (.0\dot{5} - \dot{0}\dot{5}.)^2 + (.\dot{5}\dot{0} - .5) \right\} + .0\dot{0}\dot{5}.$$

5. Prove that the limit of $.0\dot{3}\dot{5}$ is $\frac{7}{198}$.

6. Add together the two following infinite series:—

$$\frac{9}{10^2} + \frac{9}{10^4} + \frac{9}{10^6} + \ldots.$$

$$\frac{909}{10^3} + \frac{909}{10^7} + \frac{909}{10^{11}} + \ldots.$$

7. A moving body passes over in the first hour a space of $4\frac{1}{2}$ miles, in the second hour one-tenth of this, in the third hour one-tenth of the space traversed in the second hour, and so on. Find the total distance it will traverse if it go on for ever.

DIFFERENT WAYS OF EXPRESSING THE SAME MAGNITUDE.

(*Continued from p.* 124.)

125. Nothing need now be added to the knowledge which the learner must already possess regarding the various modes of expressing the same quantity, and the means by which we pass from one form to another. Attention, however, is here again called to the subject because, in dealing with it before, if fractions occurred we could use only the ordinary fractional form; whereas similar exercises with the general decimal notation are of greater practical importance. Such exercises are now given, the same arrangement being followed as before.

126. First, when the quantity as given is expressed in terms of a single unit.

Example 1. Express £2.68125 (1) in terms of the *shilling*, (2) in terms of the *penny*, and (3) in terms of the usual variety of units.

(1) $$\begin{aligned}\text{£}2.68125 &= (2.68125 \times 20) \text{ shillings}\\ &= 53.625 \text{ shillings.}\end{aligned}$$

(2) $$\begin{aligned}\text{£}2.68125 &= (2.68125 \quad 20 \times 12) \text{ pence}\\ &= 643.5 \text{ pence.}\end{aligned}$$

(3) $$\begin{aligned}\text{£}2.68125 &= \text{£}2 \text{ and } (.68125 \times 20) \text{ shillings}\\ &= \text{£}2 \text{ and } 13.625 \text{ shillings}\\ &= \text{£}2\ 13\text{s. and } (.625 \times 12) \text{ pence}\\ &= \text{£}2\ 13\text{s. } 7.5\text{d.}\\ &= \text{£}2\ 13\text{s. } 7\tfrac{1}{2}\text{d.}\end{aligned}$$

Instead of the self-explanatory process in (3) we may in practice arrange the necessary calculation shortly as follows:—

$$\begin{array}{r} £2.68125 \\ 20 \\ \hline 13.62500 \\ 12 \\ \hline 7.500 \\ 4 \\ \hline 2.0 \end{array}$$

Example 2. Express .81875 yd. (1) in terms of the *furlong*, (2) in terms of the *mile*, (3) in terms of the usual variety of units.

(1) $.81875 \text{ yd.} = \dfrac{.81875}{5\frac{1}{2} \times 40} \text{ fur.}$

$= \dfrac{.81875}{220} \text{ fur.}$

$= .0037215\dot{9}\dot{0} \text{ fur.}$

(2) $.81875 \text{ yd.} = \dfrac{.81875}{1760} \text{ mi.}$

$= .0004651988\dot{6}\dot{3}.$

(3) $.81875 \text{ yd.} = (.81875 \times 3) \text{ ft.} = 2.45625 \text{ ft.}$

$= 2 \text{ ft.} + (.45625 \times 12) \text{ in.}$

$= 2 \text{ ft.} + 5.475 \text{ in.}$

$= 2 \text{ ft. } 5\frac{18}{40} \text{ in.}$

Example 3. Express $£.188541\dot{6}$ in the usual variety of units.

$$\begin{array}{r} £.188541\dot{6} \\ 20 \\ \hline 3.77083\dot{3}\text{s.} \\ 12 \\ \hline 9.24999\dot{9}\text{d.} \\ \textit{i.e. } 9.25\text{d.} \\ 4 \\ \hline 1.00\text{f.} \end{array}$$

Result, $£.188541\dot{6} = 3\text{s. } 9\frac{1}{4}\text{d.}$

When the decimal fraction, as in the last example, is periodic, there is sometimes a saving of labour in changing it into the ordinary fractional form.

Exercises. Set LXXII.

Express—

1. .2178d. and .0127 guin. in terms of the *pound*.
2. .6042s. and £.6042 in terms of the *crown*.
3. .01344 da. in terms (1) of the *minute*, (2) of the *week*.
4. .26901 st. in terms (1) of the *hundredweight*, (2) of the *ounce*.
5. .01111 yd. in terms (1) of the *mile*, (2) of the *inch*.
6. $.31\dot{4}2\dot{8}$ sq. po. in terms (1) of the *square foot*, (2) of the *acre*.

Express in the usual way by means of integral numbers and lower units—

7. £.3625.	8. £2.778125.
9. £.040625.	10. £1.470385.
11. 3.0525 hr.	12. 1.178125 ton.
13. .71425 yd.	14. .91875 ac.
15. 1.875 gall.	16. 1.96875 cub. yd.
17. £$1.81041\dot{6}$.	18. $3.30\dot{3}57142\dot{8}$ cwt.
19. $.010\dot{8}0246913\dot{5}$ sq. yd.	20. $1.6647\dot{7}\dot{2}$ ac.

127. Secondly, when the quantity as given is expressed in terms of the usual variety of units.

Example 1. Express £6 12s. $2\frac{3}{4}$d. in terms of the *pound* alone.

$$\tfrac{3}{4}\text{d.} = .75\text{d.}$$
$$\therefore 2\tfrac{3}{4}\text{d.} = 2.75\text{d.} = \frac{2.75}{12}\text{s.}$$
$$= .2291\dot{6}\text{s.}$$
$$\therefore 12\text{s. } 2\tfrac{3}{4}\text{d.} = 12.2291\dot{6}\text{s.} = £\frac{12.2291\dot{6}}{20}$$
$$= £.611458\tfrac{1}{3}$$
$$\therefore £6\ 12\text{s. } 2\tfrac{3}{4}\text{d.} = £6.611458\tfrac{1}{3}.$$

In practice this process is arranged as follows :—

$$\begin{array}{r|l} 4 & 3\text{f.} \\ 12 & 2.75\text{d.} \\ 20 & 12.2291\dot{6}\text{s.} \\ & £6.611458\tfrac{1}{3} \end{array}$$

After dividing by 4, we prefix the 2d., and so on.

It will be seen that the mode of proceeding given before § 108, is not so short as this.

Example 2. Express 5 mi. 2 fur. 3 yd. 2 ft. in terms of the *pole* alone.

$$\begin{array}{rl} 5 & \text{mi. } 2 \text{ fur.} \\ 8 & \\ \hline 42 & \text{fur.} \\ 40 & \\ \hline 1680 & \text{po.} \end{array} \qquad\qquad \begin{array}{r} 3)\underline{2 \text{ ft.}} \\ 3.\dot{6} \text{ yd.} \\ \underline{2} \\ 11)\underline{7.\dot{3}} \\ .\dot{6} \text{ po.} \end{array}$$

$$\begin{aligned} \text{Thus } 5 \text{ mi. } 2 \text{ fur.} &= 1680 \text{ po.} \\ \text{and } 3 \text{ yd. } 2 \text{ ft.} &= .\dot{6} \text{ po.} \end{aligned}$$

Therefore the required result $=1680.\dot{6}$ po.

EXERCISES. SET LXXIII.

Using decimal fractions, express

1. £1 16s., 12s. 9d., and £1 6s. 3d. in terms of the *pound* only.
2. £12 17s. $5\frac{1}{4}$d., £2 0s. $2\frac{1}{4}$d., and £6 1s. $6\frac{1}{2}$d. in terms of the *pound* only.
3. 4 cwt. 1 qr. 7 lb. in terms of the *ton*, and 4 oz. 7 dr. in terms of the *pound*.
4. 2 tons 21 lb. in terms of the *hundredweight*, and 5 lb. 3 oz. 9 dr. in terms of the *stone*.
5. 17 hr. 33 min. in terms of the *day*; 6 hr. 17 sec. in terms of the *minute*.
6. 13 ft. 9 in. in terms of the *yard*, and 10 fur. 37 po. 3 yd. in terms of the *mile*.
7. 17 yd. 2 ft. 6 in. in terms of the *fathom*, and 276 yd. 2 ft. in terms of the *mile*.
8. 2 ro. 17 sq. po., 1 ro. 22 sq. yd., and 2632 sq. yd. 6 sq. ft. in terms of the *acre*.
9. 11 cub. ft. 1593 cub. in. in terms of the *cubic yard*, and 7 bus. 3 gall. in terms of the *quarter*.
10. 1 qr. 24 lb. $9\frac{3}{8}$ oz. in terms of the *hundredweight*, and 16 fathoms $5\frac{3}{8}$ ft. in terms of the *yard*.

128. When, thirdly, two quantities of the same kind are given, each expressed in terms of one or more units, and the requirement is to express the one in terms of the other, we may proceed exactly as before.

Example. Express £1 6s. $5\frac{1}{4}$d. in terms of 4s. 2d. (the United States dollar).

$$\left.\begin{array}{lll} £1 & 6\text{s.} & 5\tfrac{1}{4}\text{d.} = 1269 \text{ far.} \\ & 4\text{s.} & 2\text{d.} \;\; = 200 \text{ far.} \end{array}\right\}$$

$$\begin{aligned} \therefore\ £1\ \ 6\text{s.}\ \ 5\tfrac{1}{4}\text{d.} &= \tfrac{1269}{200} \text{ dol.} \\ &= 6.345 \text{ dol.} \end{aligned}$$

Exercises. Set LXXIV.

1. Express £7 1s. 9d. and £1 9s. $8\frac{1}{4}$d. as decimal fractions of £10.
2. Express 3s. $2\frac{1}{2}$d. as a decimal fraction of £10 10s.
3. What decimal fraction of £1 11s. 6d. is £1 3s. 3d.?
4. Express 1 cwt. $3\frac{1}{2}$ lb. as a decimal fraction of 12 stone.
5. What decimal fraction is 5 hr. $9\frac{1}{11}$ min. of $2\frac{1}{8}$ da.?
6. Express 1565 rupees in terms of the *franc*, taking 1 rupee equal to 2s. 3d. and 1 franc equal to $9\frac{3}{4}$d.

DECIMAL SYSTEMS OF UNITS OF MEASUREMENT.

129. The advantage of having our units of measurement so related to each other that any one is contained *ten* times in the next higher unit of the same kind has been already referred to, and must be more abundantly manifest now that the extension of the decimal notation to fractional numbers is known and thoroughly understood. By scientific men it has long been recognised, and in consequence of their endeavours *decimal* systems of units of measurement have already been established in many countries; and it is not too much to expect that they will yet become perfectly general throughout the world.

In Britain the old systems still hold their ground; but much has been done to prepare the way for superseding them, so that a knowledge of the systems likely to be adopted is desirable.

I.—PROPOSED DECIMAL SYSTEM OF BRITISH MONEY.

130. Various decimal systems of money have been proposed, those being most in favour in which one of the present units is retained as the standard unit, so as to make the transition easy from the old system to the new.

The scheme which at present seems most likely to be adopted, if any change should be made, is that which was recommended after much consideration by a committee of the House of Commons in 1853. In it the standard unit is the present POUND sterling, the other three being, of course, a *tenth*, a *hundredth*, and a *thousandth* of this. For the second unit the name FLORIN is in use; for the third and fourth no names have been fixed, but CENT and MIL seem likely to be very generally acceptable. The fourth being £$\frac{1}{1000}$ would, however, differ so little from a farthing, which is £$\frac{1}{960}$, that the name *farthing* might be transferred to the new unit. Instead of the table on p. 42, we should then have the following:—

10 mils = 1 cent
10 cents = 1 florin
10 florins = 1 pound.

131. With such a system of units the various possible ways of expressing the same quantity are immediately evident, instead of requiring to be found by the tedious processes necessary in the case of the present units (see pp. 45—60). For example:

£21 6fl. 3c. 2m. = £21.632
= 216.32 fl.
= 2163.2 c.
= 21632 m.

£3 4 m. = 3004 m.
= 30.04 fl.

and 76305 m. = £76 3 fl. 5 m.

Further, it is worthy of remark that in all probability we should no longer use more than *one* unit in expressing a quantity in writing. Thus

£21.632

would be written, instead of

£21 6 fl. 3c. 2 m.

and so on.

132. As a consequence of this it would not be necessary to make special practice, as we have done, of the addition, subtraction, multiplication, and division of sums of money, &c.; that is to say, there would be nothing corresponding to the tedious work found in the present text-book, extending from p. 52 to p. 75.

Example 1. A man's yearly income is £250 5 fl., and his expenditure £115 6 fl. 2c. 5 m. What does he save?

Sum saved $= £250.5$	or	$= 250500$ mils	
-115.625		-115625	„
$= £134.875$		$= 134875$	„

Answer, £134 8 fl. 7 c. 5 m.

Example 2. What is the cost of $15\frac{1}{2}$ yards of cloth at 3 florins 6 mils per yard?

$$\begin{aligned}\text{Cost} &= 306 \text{ mils} \times 15\tfrac{1}{2}\\ &= 4743 \text{ mils.}\\ &= £4\ 7 \text{ fl. } 4 \text{ c. } 3 \text{ m.}\end{aligned}$$

EXERCISES. SET LXXV.

1. Express in *mils* £3, £6 5 fl., £2 3 fl. 4 c. 3 m., 8 fl. 2 m., and £5 $5\frac{1}{2}$ c.
2. Express in terms of the *pound* 6 fl. 1 c., 5 c. 3 m., and 32 fl. 8 m.
3. Express in terms of the *florin* £5, 5 cents, 5 mils, and £13 2 fl. 3 m.
4. Find the sum of £113 2 fl., £2 5 c., 3 fl. 6 c. 4 m., and 13 fl.
5. Express in mils the sum of £19, 19 fl., 19 c., and 19 m.
6. Subtract £2 3 c. from £12 5 fl. 4 m.
7. Express in florins the difference between £50 and 50 cents.
8. Find the cost of 306 articles at 3 fl. 6 m. each.
9. 25 lb. of tea are bought for £5 6 fl. What does it cost per lb.?
10. What will 580 eggs cost at 5 c. 4 m. per doz.?
11. 18 cwt. of coal cost 7 fl. 5 c. What would 37 tons cost at this rate?
12. 16 lb. of sugar cost 3 fl. 2 c. How much may be bought for 7 c. 5 m.?
13. How many articles, each worth 3 fl. 2 c. 6 m., may be bought for £24 1 fl. 2 c. 4 m.?

133. A sum of money being given in terms of our present units, we are able to find the expression for it in the new system by following the process of § 127. What is wanted is simply that the portion of it stated in shillings and pence and farthings be expressed as *a decimal fraction of a pound.*

Example 1. What are the equivalents of £1 14s. 6d. and £7 0s. 4¼d. in the proposed decimal system?

(1)
$$\begin{array}{r} 12\underline{)6d.} \\ 20\underline{)14.5s.} \\ £1.725 \end{array}$$

(2)
$$\begin{array}{r} 4\underline{)1 f.} \\ 12\underline{)4.25d.} \\ 20\underline{)\ .35416\dot{6}s.} \\ £7.0177083\dot{3}. \end{array}$$

Answer, £1 7 fl. 2 c. 5 m., and £7 1 c. 1 m. nearly.

In the event of the adoption of the proposed system, this operation would be very common, so that any speedier mode of arriving at the result should be practised. Thus we might easily bear in remembrance that

2s.	=	£.1
1s.	=	.05
6d.	=	.025
¼d.	=	.001 nearly,

and proceed as in the following instances.

Example 2. Express 14s. 8¼d. and £7 5s. 3½d. in accordance with the proposed system.

(1)	14s. (being 7 florins)	=	£.7
	6d.	=	.025
	2¼d. (being 9 farthings)	=	.009...
	∴ 14s. 8¼d.	=	£.734...
(2)	5s. (being 2½ florins)	=	£.25
	3½d. (being 14 farthings)	=	.014...
	∴ £7 5s. 3½d.	=	£7.264...

Exercises. Set LXXVI.

Find the equivalents in the proposed decimal system of—

1. 12s.	2. 13s.	3. 13s. 6d.
4. 17s. 6d.	5. 6s. 6d.	6. £1 8 0.

7. £3 7 6.	8. £2 0 6¾.	9. £1 1 0.
10. £1 16 2½.	11. £2 13 4¼.	12. £5 5 5¼.
13. £4 3 8½.	14. £1 7 10¼.	15. £2 11 9.
16. £10 10 7.	17. £7 17 6¾.	18. £11 11 11½.
19. £10 1 6¼.	20. £12 1 0¼.	21. £20 0 6½.

134. Conversely, a sum of money being given in terms of the proposed units we can find the expression for it in the present system by the method followed in § 126.

Example 1. What are the equivalents of £7 6fl. 3c. 2m. and £2 5c. in the present money system?

$$
\begin{array}{r}
(1)\quad £7.632 \\
20 \\ \hline
12.640 \\
12 \\ \hline
7.68 \\
4 \\ \hline
2.72
\end{array}
\qquad\qquad
\begin{array}{r}
(2)\quad £2.05 \\
20 \\ \hline
1.00
\end{array}
$$

Ans. £7 12s. 7¾d., nearly, and £2 1s.

Here again, however, if we remember the four facts given in the preceding paragraph, but now rather in the form—

1 florin, or £.1 = 2s.
5 cents, or £.05 = 1s.
25 mils, or £.025 = 24f.
1 mil, or £.001 = 1f., nearly,

we may with ease at once write down a result sufficiently accurate for almost all purposes.

Example 2. Express £16.376 and £1.788 in accordance with the present money system.

(1) £16.3	=	£16	6	0
.05	=	0	1	0
.026 = 25f., nearly,	=	0	0	6¼, nearly.
∴ £16.376	=	16	7	6¼, nearly.
(2) £1.75	=	1	15	0
.038 = 37f., nearly,	=	0	0	9¼, nearly.
∴ £1.788	=	1	15	9¼, nearly.

EXERCISES. SET LXXVII.

Find the equivalents in the present money system of—

1. 6 fl. 5 c.	2. 8 fl. 5 c.	3. 1 fl. 2 c. 5 m.
4. 3 fl. 2 c. 5 m.	5. 6 fl. 7 c. 5 m.	6. 9 fl. 7 c. 5 m.
7. £8 2 c. 5 m.	8. £3 5 c.	9. £6 2 fl. 7 c. 5 m.
10. £3.175.	11. £6.387.	12. £4.415.
13. £0.634.	14. £1.078.	15. £2.707.
16. 5750 m.	17. 21775 c.	18. 22.1875 c.
19. 631.25 m.	20. 3 fl. 6 c. 2½ m.	21. 9 fl. 6 c. 5.625 m.

II.—PROPOSED DECIMAL SYSTEM OF UNITS OF SPACE AND MASS.

135. The decimal system of units of space and mass which is likely to be adopted is that originated by the French and generally known as the *metric system*, from the name of its fundamental unit, the *metre*. In France, as in other countries, there had been in use, for a lengthened period, a confusing variety of units, the origin, of course, of much inconvenience and often of fraud. But soon after the revolution of 1789 the National Assembly took up the matter in earnest, and the result was the production of a thorough system, decimal in character, scientifically built upon one fundamental unit, entirely new in nature and nomenclature, and thus adapted to the wants of any and every nation as well as the French. This system was introduced in 1795. It made way at first with much difficulty, but now has been adopted by several other countries on the continent of Europe, and is fast spreading. In Britain its use was legalised in 1864, and already many scientific men employ it.

136. UNITS OF LENGTH.—The French reformers aimed at securing a permanent basis in nature for their fundamental unit, the unit of length, as already was the case with the units of time. For this purpose they selected the distance measured on the surface of the sea, from the north pole to the equator, the ten-millionth part of which was taken as the unit and designated a *Metre*. It is not, how-

ever, defined in this way. A platinum rod was prepared to represent the required length when at the temperature of melting ice; and it is this rod kept in Paris which fixes the length of the metre throughout the world: it may not be accurately the ten-millionth part of the distance referred to. Of the other units the smaller are $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, of a metre, named respectively the *decimetre*, *centimetre*, *millimetre*, from the Latin words for "ten," "hundred," "thousand"; and the larger are 10, 100, 1000, 10000 metres, named respectively the *decametre*, *hectometre*, *kilometre*, *myriametre*, from the Greek words for "ten," "hundred," "thousand," "ten thousand"; the decametre, hectometre, myriametre are, however, seldom spoken of. Thus

$$1634.216 \text{ metres} = \left\{ \begin{array}{l} .1634216 \text{ myriam.,} \\ 1.634216 \text{ kilom.,} \\ 16.34216 \text{ hectom.,} \\ 163.4216 \text{ decam.,} \\ 16342.16 \text{ decim.,} \\ 163421.6 \text{ centim.,} \\ 1634216 \text{ millim.,} \end{array} \right.$$

or, again, = 1 kilom. 634 metres 216 millim.

For a knowledge of the relative magnitude of these units and the present British units it is sufficient to note that

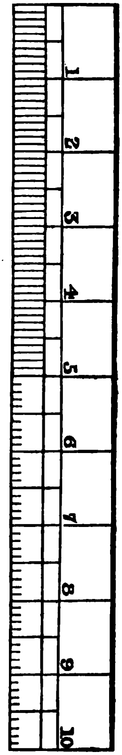

Fig. 2.
A Decimetre Measure.
Actual size.

1 metre = 39.37079 in.	1 ft. = 3.0479 decimetres.
1 kilometre = .62138 mi.	1 mi. = 1.60931 kilometre.

But it is also worth remembering that 12 yards are nearly 11 metres, that 1 decimetre is nearly 4 inches, and that 1 kilometre is approximately $\frac{5}{8}$ of a mile. A tolerably accurate representation of a decimetre measure subdivided into centimetres and millimetres is annexed.

Exercises. Set LXXVIII.

1. Express 3600 metres in terms (1) of the kilometre, (2) of the decimetre, (3) of the millimetre.

2. Express 156 decimetres in terms (1) of the kilometre, (2) of the millimetre.

3. Express 1134 millimetres in decimetres and millimetres.

4. Express 4.62 metres in decimetres and centimetres.

5. Express 3105600 centimetres in kilometres and millimetres.

6. How often is a millimetre contained in a kilometre, and a centimetre in a decametre?

7. Express 17 kilometres 16 metres (1) in decimetres, (2) in decametres.

8. Express 4 decimetres 3 millimetres in terms (1) of the metre, (2) of the millimetre.

Find the equivalents in the metric system of—

9. 7 yd.	3 yd. 1 ft. 6 in.	10 in.
10. $16\frac{1}{2}$ mi.	4 mi. 2 fur.	5 mi. 100 yd.

11. Express $10\frac{1}{2}$ metres in yards, &c., and 74 kilometres in miles and yards.

12. A town is situated in 9^c north latitude. How far is it from the equator?

13. How far from the North Pole is a town situated in lat. 22° 30′ N.?

137. Units of Area.—In the measurement of land and other large surfaces the principal unit is the ARE, which is equal to 1 square decametre, or a square whose side is 10 metres long, the other units in use being a larger one the *hectare* (= 100 ares) and a smaller one the *centiare* (= $\frac{1}{100}$ are).

Just as we, however, in the measurement of small surfaces use the square yard, square foot, &c., so the French use the *square metre, square decimetre, square centimetre, square millimetre.* The multiplier connecting these is of course not 10 but 100, for 1 metre being equal to 10 decimetres it follows that 1 square metre is equal to 100 square decimetres, and so on. The full table of units therefore is—

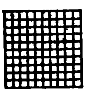

Fig. 3. A Square Centimetre subdivided into 100 Sq. Millim.

1 hectare = 100 ares.	1 sq. metre = 100 sq. decim.
1 are = 100 centiares, or square metres.	1 sq. decim. = 100 sq. centim.
	1 sq. centim. = 100 sq. millim.

Thus,

5.1314 hectares = 5 hectares 13 ares 14 centiares;
4.3821 sq. metres = 4 sq. metres 38 sq. decim. 21 sq. centim.;
and
5.163 sq. decim. = 5 sq. decim. 16 sq. centim. 30 sq. millim.

For the comparison of these units with the corresponding British units, it suffices to note that

1 sq. metre	= 1.196033 sq. yd.	1 sq. yd.	= .836097 sq. metre.
1 hectare	= 2.471143 ac.	1 ac.	= 40.4671 ares;
			= $\frac{2}{5}$ hectare, nearly.

Exercises. Set LXXIX.

1. Express 1.143 hectare in ares and centiares.

2. Express 141100 sq. centimetres (1) in terms of the are, (2) in sq. metres and sq. decimetres.

3. Express 2.73 hectares, 146 ares, and 17000 sq. decimetres each in terms of the sq. metre.

4. How many sq. decimetres are there in an are, and how many sq. millimetres in a centiare?

Find the equivalents in the metric system of—

5. $4\frac{1}{2}$ sq. yd.	100 sq. ft.	13 sq. ft. 40 sq. in.
6. $\frac{3}{5}$ ac.	16 ac. 3 ro.	2 ac. 2 ro. 16 sq. po.

7. Express 1.5 sq. decimetre in square inches, and 5.62 hectares in acres, &c.

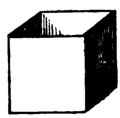

Fig. 4. A Cubic Centimetre. Actual size.

138. Units of Volume.—(I.) For the purposes for which we at present use the cubic yard, cubic foot, and cubic inch, the French use the *cubic metre*, *cubic decimetre*, *cubic centimetre*, and *cubic millimetre*, any one of which must evidently contain 1000 times the one which follows it. Thus

3.066117 cub. metres = 3 cub. metres 66 cub. decim. 117 cub. centim.

and

.31425 cub. decim. = 314 cub. centim. 250 cub. millim.

(II.) For the measurement of liquids, grain, &c., the principal unit (of *capacity*, as it is called) is the *litre*, which is equivalent to a cubic decimetre. Of the other units in

use the smaller are the *decilitre* ($=\frac{1}{10}$ litre) and *centilitre* ($=\frac{1}{100}$ litre); and the larger the *decalitre* (=10 litres) and *hectolitre* (=100 litres).

For the comparison of these units with the present British units of volume it is to be noted that

1 cub. decim. = 61.02703 c. in.		1 cub. ft. = 28.315103 c. decim.	
1 litre	= 1.76077 pt.	1 gall.	= 4.543456 litres.
1 hectolitre	= 2.75121 bus.	1 qr.	= 2.907813 hectol.

EXERCISES. SET LXXX.

1. Express 117.64 cub. decimetres in terms (1) of the cub. centimetre, (2) of the cub. metre.

2. Express .01743 cub. metre and 31462000 cub. millimetres in cub. decimetres and cub. centimetres.

3. How often is a cub. millimetre contained in a cub. decimetre?

4. Express .3104 hectolitre (1) in terms of the centilitre, (2) in litres and centilitres.

5. Express 3.142 litres in terms (1) of the hectolitre, (2) of the millilitre.

6. How many cub. centimetres are there in 3 litres, and how many cub. decimetres in 3 hectolitres?

7. Express 1.53 litre in terms of the cub. centimetre, and 4163 hectolitres in terms of the cub. metre.

Find the equivalents in the metric system of—

8. 144 cub. in.	20 cub. yd. 13 cub. ft.	$10\frac{1}{2}$ gall.
9. 7 qr. 3 bus.	10 bus. 1 pk.	3 quarts 1 pt.

10. Express 275 cub. centimetres in cub. inches, and 10 cub. metres in terms of the cub. yard.

11. Express 15 litres in gallons, &c., and 6 hectolitres in quarters and bushels.

139. UNITS OF MASS OR WEIGHT.—The principal unit of mass is the *gram* (French, *gramme*), which is defined to be the mass of a cubic centimetre (see Fig. 4, p. 163) of pure water at its greatest density. The smaller units are the *decigram*, *centigram*, *milligram*; the larger are the *decagram*, *hectogram*, *kilogram*. The gram being very small the last and largest of these, the kilogram, has come to be in usage the principal unit; and as such it is worthy of notice that it is the mass of a litre of pure water at its greatest density. Even it, however, is inconveniently small

Fig. 5. A Gram Weight in copper. Actual size.

for the measurement of such masses as constantly require to be mentioned in commerce, so that the *metric quintal*, equal to 100 kilograms, and the *millier* or *metric ton*, equal to 10 quintals, are also employed.

For comparison with the corresponding British units we have to note that

1 gram = 15.432348 grains,	1 grain = 64.799 milligrams.
1 kilog. = 2.204621 lb.	1 lb. = .453593 kilog.
1 quintal = 1.968412 cwt.	1 ton = 1.016047 metric ton.

EXERCISES. SET LXXXI.

1. Express 51.7 kilog. in terms (1) of the gram, (2) of the quintal, and (3) of the milligram.

2. Express .00401 kilog. and 310.4 decigrams in grams and milligrams.

3. Express 17.009 metric tons, .04 quintal, and 10000000 milligrams in kilograms.

4. How many centigrams are there (1) in a quintal, (2) in $3\frac{1}{2}$ kilog.?

5. What would be the weight of 14.3 cub. centim., of 30 cub. millim., and of .016 cub. metre of pure water at maximum density?

6. How much pure water at maximum density will weigh 316 centigrams, 4 quintals, 14 decagrams, 1 metric ton?

Find the equivalents in the Metric System of—

7. $7\frac{1}{2}$ lb.; 10 stones; 5 lb. 4 oz.

8. 2 qr. 20 lb.; 1 ton 15 cwt.; 4 cwt. 3 qr.

9. 1 cwt. 1 qr. 14 lb.; 20 tons 14 cwt. 2 qr.; 1 lb. troy.

10. Express 7 kilog. in pounds and ounces; 20 metric tons in tons; and 160 kilog. in cwt. qr. lb.

140. The foregoing account of the Metric System is of course incomplete, relating only to those portions of it at present considered likely to come into general use. Besides these, there are included in it a system of money, and a set of units for the measurement of angular space.

The monetary system has been very fairly successful, being now prevalent in France, Belgium, Switzerland, and Italy. The units used are the FRANC, which is the principal unit, and approximately equal to 10d. sterling, and the *centime* or hundredth part of a franc. The link of connection with the other units of the system is found in the

size and weight of the coins employed: for example, the bronze piece of 10 centimes, corresponding to our penny-piece, weighs 10 grams; the five-centime piece weighs 5 grams, the two-centime piece 2 grams, the one-centime piece 1 gram; and their diameters are respectively 30, 25, 20, and 15 millimetres.

The proposed units of angular measurement have unfortunately not prevailed even in France itself.

141. The superiority of the Metric System may be said to originate as follows:—

Firstly, in the choice of the fundamental unit, which supplies us with the important fact that a quadrant of the earth is approximately 10000000 metres in length.

Secondly, and chiefly, in the way in which one unit is related to the next higher unit of the same set—that is to say, in being a *decimal* system.

Example. 18 litres of wine are bought for 18.54 francs. What is the cost of 7 hectolitres at this rate?

	The price of	18 litres	= 18.54 francs
∴	„	1 litre	= 18.54 fr. ÷ 18 = 1.03 fr.
and ∴	„	7 hectol.	= 103 fr. × 7 = 721 fr.

With this may be compared the examples in § 66.

Thirdly, in the way in which the units of one set are related to those of another. Thus, from the mode of defining the units of weight, we can tell at once the weight of any volume of water at its greatest density, and therefore the weight of any volume of any substance whatever when its density (or so-called "specific gravity") with reference to water is known.

Example. Platinum is 22 times heavier than water. What is the weight of 3.5 cub. decim. of platinum?

The weight of 3.5 cub. decim. of water = 3.5 kilog.
∴ „ „ „ platinum = 3.5 kilog. × 22
= 77 kilog.

Were the question, on the other hand, "What is the weight

in *ounces* of 3.5 cub. *inches* of platinum?" we should proceed as follows, the first line being the statement of a fact requiring to be remembered separately—

	The weight of 1 cub. in. of water			$= 6.94$ oz.
$\therefore$	"	3.5 "	"	$= 6.94$ oz. $\times 3.5$
				$= 24.29$ oz.
and $\therefore$	"	"	" platinum	$= 24.29$ oz. $\times 22$
				$= 534.38$ oz.

Exercises. Set LXXXII.

1. Express the sum of 17.213 kilog., 1874.7 kilog., 3162.135 kilog., and 45.912 kilog. in quintals.

2. Express in millilitres the difference between a decilitre and the eighth part of a litre.

3. Multiply 1.375 centilitres by 16000 and state the result in hectolitres.

4. How often are 13 cub. centim. contained in 1.82 cub. metre?

5. How many phials capable of holding 2.5 centilitres may be filled from a bottle containing a litre and a quarter?

6. Find the cost of 64 metres of cloth at 2 florins 5 cents per metre.

7. 6 kilog. of sugar cost 7 cents 5 mils. What is the cost per quintal?

8. The rate of freight in a certain vessel is £.375 per quintal. What would be charged for the transport by it of 176 metric tons of merchandise?

9. How many 10-centime pieces would serve a shopkeeper instead of a kilogram weight?

10. The length of a sheet of paper is measured by laying 16 5-centime pieces edge to edge in a line. How long is the sheet?

11. Find the weight of 61 litres of wine whose specific gravity is .992.

12. A ball of pure sulphur (specific gravity, 2.033) weighs 325.28 kilog. Determine its volume.

Exercises. Set LXXXIII.

Miscellaneous.

1. Express as a decimal fraction the difference between $\frac{4}{7}$ and $\frac{5}{8}$.

2. A person starting on a journey has £17.54, and on his return he finds that the receipts marked in his note-book during the journey are £3.62, £.055, £8.475, and the expenses £4.25, £7.215, £4.16, and that he has on hand £12.605. What mistake has he made?

3. The period of the earth's revolution round the sun is 365.2422414 days. If it be taken at 365 days, what will the error amount to in 4 years? If it be taken at 365¼, what will the error amount to in 4 centuries?

4. Express .2422414 days in hours, minutes, and seconds.

5. A kilogram of pure gold is worth £134.07. At this rate what

would be the total value of three bars of gold weighing respectively 3.142 kilog., 1.006 kilog., 2.302 kilog.?

6. Express $\dfrac{1}{2+\dfrac{1}{2+\dfrac{1}{2}}}$ as a decimal fraction.

7. Simplify the expression

$$(.301 - .019462 + 4.076) + (7.2 \times .004 \times .01)$$

giving the result correct to three right-hand places.

8. The two frigid zones occupy .0826 of the earth's surface and the torrid zone .398 of it. What portion is occupied by each of the temperate zones?

9. Express £.001 + .01d. in terms of the farthing.

10. Find a decimal fraction within a millionth of the reciprocal of 111.

11. What is the cost of an iron chain 300.48 yd. long and weighing 10.45 lb. per yd., at the rate of £.056 per lb.?

12. The carriage of 17500 cwt. between two places is £49.77. What is the rate per ton?

13. Express the reciprocal of $5^6 \times 2^8$ as a decimal fraction.

14. An average height of the barometer is 28.84 inches. Express this in millimetres.

15. The planet Jupiter completes his revolution round the sun in 4332.585 days and Saturn in 10759.22 days. Express these periods in years, days, hours, &c.

16. Simplify the expression—

$$(.4006 + 12.034 - 10.00101) \times (.024 + .00003).$$

17. The carriage of 17500 cwt. a distance of 400 miles is £57.4. What is the rate per cwt. per mile?

18. What weight of wine is contained in 5000 bottles, each of which weighs 1.15 kilog., and when empty .2075 kilog.?

19. Express in terms of the ton the sum of 14.325 cwt. 713.3 lb. and 16.85 stones.

20. Find the reciprocal of the difference between $.\dot{2}\dot{7}$ and $.\dot{2}8571\dot{4}$.

21. There are 277.274 cub. in. in an imperial gallon. Find the capacity in cubic inches of a half-pint measure.

22. A regiment is on the march for 5 hours. During the first two hours the rate is 3.75 kilometres per hour; afterwards it is .225 kilom. less per hour. What is the whole distance travelled?

23. A merchant imports a quantity of material for which he pays £64.32; the carriage of it costs him £2.605 more, and the custom-house duties upon it are £.75. For what must he sell it in order to gain £5.5?

24. Simplify the expression—

$$(3.1\dot{7} - .4\dot{3}0\dot{9}) - (2.\dot{1}6 - 1.9\dot{5}46\dot{3}).$$

25. Express in pence the difference between £3.325 and $3.\dot{1}4285\dot{7}$ guineas.

26. In a parish there are 400 householders who pay school rates. 17 of them pay £.55 each, 23 pay £.46 each, 84 pay £.275 each, and

the remainder £.055 each. What income does the school board thus derive?

27. Find the fourth power of $\dfrac{1}{\frac{28}{9} - 3.\dot{1}\dot{0}}$.

28. The smaller of two numbers is 1.0078, and their difference is .00316. What is their product?

29. Seven-elevenths of a piece of cloth $14\frac{3}{4}$ yd. long are sold. Find to within less than a thousandth of a yard how much remains.

30. £6 9s. 4d. is exchanged for 162.475 francs. What would be received for £1 at this rate?

31. What would the carriage of 746.375 cwt. be at the rate of £.3 per 100 lb.?

32. Change the fractions in the expression

$$(3\tfrac{3}{4} + 2\tfrac{7}{16} - 1\tfrac{1}{20}) \times 3\tfrac{1}{40}$$

into decimal fractions, and then perform the operations indicated. Perform the operations also without making the change referred to, and examine whether the two results agree.

33. A porter who cannot carry more than 50.4 kilog. is engaged to remove 8.064 quintals of fruit a distance of 1.375 kilometres. What distance must he travel before completing his work?

34. If the postage for letters weighing not more than an ounce were changed from 1d. to 4 mils., what would be the loss to the national revenue on every million of such letters?

35. Simplify the expression—

$$(2.7\dot{8} \times 3.\dot{0}\dot{9}) \div (3.\dot{0}1 - 2.\dot{1}\dot{0}).$$

36. Perform the operations indicated in the expression—

$$(2\tfrac{2}{3} + 1\tfrac{1}{60}) \div (\tfrac{4}{25} - \tfrac{1}{8}),$$

(1) after expressing the given fractions as decimal fractions, (2) without doing this.

37. A coin weighs .675 oz., and .64 of it is copper. How much copper will be necessary in coining half a million of such pieces?

38. Perform the operations indicated in the interminate expression*—

$$\frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \frac{1}{9^4} + \ldots .$$

as far as may be necessary to obtain the sum correct to five right-hand places.

39. The length of a piece of cloth is ascertained by means of a metre rod to be 207.36 metres, but afterwards it is found that the rod is too short by 3 millimetres. What is the true length of the piece?

40. A tradesman buys 342.45 metres of cloth at 18.25 francs per metre, and sells $\frac{7}{8}$ of it at 19.4 francs per metre. At what price per metre must he sell the remainder to make a total gain of 500 francs?

41. A person after paying $\frac{1}{4}$, $\frac{2}{7}$, and $\frac{3}{11}$ of his debt still owes £1338.6. What was the full amount of his debt?

42. If cloth be bought at £.462 per yd. and sold at £.397, what gain

* Called an "infinite series."

is made (1) when the quantity bought is 100 yd., (2) when the quantity bought costs £100 ?

43. Find the limit (correct to seven right-hand places) of the infinite series—

$$1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \ldots.*$$

44. A horse is supplied with 1935 kilograms of hay, 879 kilograms of straw, and 2 hectolitres of oats in a year. Supposing the hay to cost 8.75 fr. per quintal, the straw 4.5 fr. per quintal, and the oats 1.08 fr. per decalitre, find the total cost.

45. Oranges are bought at £.475 per hundred and sold at £.048 per doz., the loss sustained being £5.4. What is the number of oranges?

46. A fraction whose numerator is 134 is equal to a decimal fraction, part of which is .446... What is the denominator ?

47. The water of a certain lake contains .045 of its weight of salt. What weight of the water would be needed in order to produce 643.23 lb. of salt ?

48. Find the limit (correct to four right-hand places) of the infinite series—

(1) $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots.$$

(2) $$1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \ldots.$$

49. Express as a single fraction the difference between the two infinite series—

$$\frac{8}{10} + \frac{8}{100} + \frac{8}{1000} + \ldots.$$

$$\text{and } \frac{25}{10^3} + \frac{25}{10^6} + \frac{25}{10^9} + \ldots..$$

50. Complete the following statistical table, designed to illustrate the density of population in the countries mentioned :—

Country.	Area in Square Miles.	Population.	No. of Acres to each Inhabitant.
Britain and Ireland .	121262	31629299	
France	204928	36102921	
Sweden and Norway	294641	6043315	
British India . . .	1453367	238830958	
New Zealand . . .	121875	332800	
Dominion of Canada	3513325	3718745	
State of New York .	47000	4382759	
State of Oregon . .	102606	90923	
Brazil	3287964	9448233	

* The dot (.) is often used, as in the denominators here, for the symbol of multiplication.

CLASSIFIED COLLECTION OF PRACTICAL EXAMPLES, WITH EXPLANATORY NOTES ON THE SUBJECTS.

142. Many examples of arithmetical questions of the kinds likely to arise in ordinary life have already been given. Those, however, referring to certain subjects, such as the price of goods, interest on money, division of profits among partners, &c., occur more frequently than others, and on this account deserve more space and a little special attention.

The answering of an arithmetical question, as the learner must have already felt, depends as well upon a knowledge of the technical terms involved in the statement of it, and a correct understanding of the subject dealt with, as upon a knowledge of arithmetic. A problem in reference to a body falling from a height may be stated in such a way as to require for its solution a knowledge of the laws of motion; and another dealing with the scoring in a cricket match might be quite beyond the power of an arithmetician who knew nothing of the game and its technical expressions. Indeed, the learner has already solved problems on several of the commercial subjects we are now approaching, which, in all likelihood, would have puzzled him had they been stated in the ordinary language peculiar to business. We shall accordingly give, in addition to illustrative examples, short notes explanatory of the subjects dealt with and definitions of the more common technical expressions; not because a text-book of arithmetic is the proper place for such things, but because without them the learner would be in a great measure debarred from having practice with examples of much commercial importance.

Further, as every arithmetical problem requires at the outset more or less time for consideration as to what opera-

tions are to be performed, we shall, where advisable, do away with the necessity for this preliminary reasoning by giving a practical rule for the mode of procedure in the case of problems of particular kinds.

143. COST OF GOODS, &c.—The most common kind of calculation in ordinary life is finding the cost of a quantity of goods when the price per unit of measurement is known. Three cases fall to be considered, as the quantity of goods may be specified in different ways.

I. When there is only one unit used and the number is integral: as, 63 tons, 112 acres, &c.

Here, as is well known, we have simply to find the result of multiplying the price of one ton by the number of tons, the price of one acre by the number of acres, and so on. No examples of this need therefore be given, except perhaps for the case where the number of units is 12, or a multiple of 12; for then there is possible a simplification not yet referred to, but well worth remembering, and which arises from the fact of the number of pence in a shilling being 12.

The price of 12 things at 1d. each is evidently 1s., the price at 2d. each is 2s., the price at 3d. each is 3s., the price at 4d. each is 4s., and so on; if therefore, in addition to this, we remember that the price at ¼d. each is 3d., the price at ½d. each is 6d., and the price at ¾d. each is 9d., we shall be able to tell with ease in almost all cases the price of 12 things when the price of 1 is given.

Example 1. What is the cost of 12 articles at 6¾d. each?

The cost of 12 articles at 6d. each = 6s.
and " " " ¾d. " = 9d.
∴ " " " 6¾d. " = 6s. 9d.

Example 2. What is the cost of 12 articles at 7½d. each? *Ans.* 7s. 6d.

Example 3. What is the cost of 12 lb. of tea at 2s. $8\frac{1}{4}$d. per lb.?

2s. $8\frac{1}{4}$d. = $32\frac{1}{4}$d., so that the answer is 32s. 3d.

Example 4. What is the cost of 4 doz. boxes of pens at $10\frac{1}{2}$d. per box?

1 doz. at $10\frac{1}{2}$d. each will cost 10s. 6d.
∴ 4 " " " " 42s.

Example 5. What is the price of 49 yd. of cloth at $10\frac{1}{4}$d. per yd.?

12 yd. at $10\frac{1}{4}$d. each will cost 10s. 3d.
∴ 48 yd. " " " 41s.
and ∴ 49 yd. " " " 41s. $10\frac{1}{4}$d.

Exercises. Set LXXXIV.

Without using figures find the cost of—

1.	12	things at	$4\frac{3}{4}$d. each.	2.	12	things at	$5\frac{1}{4}$d. each.
3.	1 doz.	"	$6\frac{1}{2}$d. "	4.	1 doz.	"	$1\frac{1}{4}$d. "
5.	1 doz.	"	1s. $0\frac{1}{2}$d. "	6.	1 doz.	"	1s. $5\frac{1}{2}$d. "
7.	1 doz.	"	2s. $1\frac{1}{4}$d. "	8.	1 doz.	"	3s. $4\frac{3}{4}$d. "
9.	1 doz.	"	6s. $7\frac{1}{4}$d. "	10.	1 doz.	"	9s. $8\frac{1}{2}$d. "
11.	1 doz.	"	12s. $1\frac{3}{4}$d. "	12.	1 doz.	"	15s. $4\frac{1}{2}$d. "
13.	2 doz.	"	$7\frac{1}{2}$d. "	14.	2 doz.	"	$8\frac{3}{4}$d. "
15.	3 doz.	"	$6\frac{1}{4}$d. "	16.	3 doz.	"	$9\frac{1}{2}$d. "
17.	4 doz.	"	1s. $0\frac{1}{4}$d. "	18.	5 doz.	"	$6\frac{1}{2}$d. "
19.	11 doz.	"	$2\frac{1}{2}$d. "	20.	9 doz.	"	1s. 1d. "
21.	24	"	$3\frac{3}{4}$d. "	22.	48	"	$4\frac{1}{4}$d. "
23.	72	"	$3\frac{1}{2}$d. "	24.	36	"	1s. $7\frac{1}{2}$d. "
25.	$2\frac{1}{2}$ doz.	"	$4\frac{1}{2}$d. "	26.	25	"	$11\frac{1}{2}$d. "
27.	109	"	$1\frac{1}{2}$d. "	28.	37	"	$1\frac{3}{4}$d. "
29.	50	"	$7\frac{1}{2}$d. "	30.	124	"	1s. $0\frac{1}{2}$d. "
31.	1 gross	"	$4\frac{1}{4}$d. "	32.	$1\frac{1}{2}$ gross	"	$10\frac{1}{2}$d. "

II. When there is only one unit used and the number is not an integer.

Example 1. Find the cost of $\frac{5}{8}$ yd. of cloth at 13s. 6d. per yard.

As a yard of the cloth costs 13s. 6d., five-eighths of a yard must cost five-eighths of 13s. 6d., or—

13s. 6d. × $\frac{5}{8}$, which = 67s. 6d. ÷ 8 = 8s. $5\frac{1}{4}$d.

Example 2. Calculate the cost of $113\frac{7}{12}$ tons of coal at £1 6s. 8d. per ton.

Here, to find the cost of 113 tons we multiply £1 6s. 8d. by 113, and for the cost of $\frac{7}{12}$ ton we take $\frac{7}{12}$ of £1 6s. 8d., and then add the two results. Thus—

$$\text{Price of } 113\tfrac{7}{12} \text{ tons} = \begin{cases} \text{£}1\ 6\text{s. } 8\text{d.} \times 113 \\ +\text{£}1\ 6\text{s. } 8\text{d.} \times \frac{7}{12} \end{cases}$$

$$= \begin{cases} \text{£}150\ 13\text{s. } 4\text{d.} \\ + \quad\quad 15\text{s. } 6\tfrac{2}{3}\text{d.} \end{cases}$$

$$= \quad \text{£}151\ 8\text{s. } 10\tfrac{2}{3}\text{d.}$$

This clearly amounts to multiplication by $113\frac{7}{12}$, and thus we see that whether the number which specifies the quantity of goods be integral or fractional the cost is got by *multiplying the price of one unit by this number.*

Exercises. Set LXXXV.

Find the cost of—

1. $\frac{1}{4}$ ton at 5s. 3d. per ton.
2. $\frac{1}{7}$ cwt. at £2 2s. 7d. per cwt.
3. $\frac{3}{8}$ yd. at 14s. 6d. per yd.
4. $\frac{7}{12}$ yd. at 15s. 6d. per yd.
5. $\frac{7}{16}$ cwt. at £1 1s. 3d. per cwt.
6. $\frac{13}{24}$ yd. at 6s. 6d. per yd.
7. $1\frac{3}{8}$ yd. at 4s. 4d. per yd.
8. $5\frac{3}{8}$ oz. at 6s. $4\frac{1}{2}$d. per oz.
9. $6\frac{3}{8}$ ac. at £12 5s. 5d. per ac.
10. $10\frac{1}{16}$ ac. at £16 5s. 5d. per ac.
11. $13\frac{1}{4}$ sq. yd. at 2s. 3d. per sq. yd.
12. $23\frac{4}{7}$ ac. at £21 0s. 7d. per ac.
13. 5.25 cwt. at £1 8s. 3d. per cwt.
14. 7.375 tons at £4 10s. 6d. per ton.
15. 4.05 lb. at 1s. 8d. per lb.
16. 13.15 lb. at 3s. 4d. per lb.
17. .0125 ton at £4 6s. 8d. per ton.
18. 6.047 lb. at 4s. $4\frac{1}{2}$d. per lb.
19. 1.416237 ton at £4 12s. 6d. per ton.
20. .70856 oz. at £4 1s. $4\frac{1}{2}$d. per oz.

III. When there is more than one unit used in specifying the quantity; as, 6 tons 14 cwt., 2 ac. 3 ro. 14 po.

Here we may proceed in the way already pointed out (§ 66), or use the following method, which will be easily understood from what has been done in § 63.

Example 1. What is the cost of 3 qr. 16 lb. of sugar at £1 17s. 6d. per cwt.?

	£	s.	d.	
	1	17	6	= price of 1 cwt.
2 qr. = $\frac{1}{2}$ of 1 cwt. ∴	0	18	9	= price of 2 qr.
1 qr. = $\frac{1}{2}$ of 2 qr. ∴	0	9	4.5	= „ 1 qr.
14 lb. = $\frac{1}{2}$ of 1 qr. ∴	0	4	8.25	= „ 14 lb.
2 lb. = $\frac{1}{7}$ of 14lb. ∴	0	0	8.03	= „ 2 lb.
∴	1	13	5.78	= price of 3 qr. 16 lb.

When, as here, fractions of a farthing occur, and the result is wanted correct only to the nearest farthing, it will generally be found advisable to use decimal fractions in the column of pence.

Example 2. Calculate the price of 13 ac. 3 ro. 18 po. of land at £35 6s. 6d. per acre.

	£	s.	d.	
	35	6	6	= price of 1 ac.
∴ mult^{g}. by			13	
we have	459	4	6	= price of 13 ac.
2 ro. = $\frac{1}{2}$ of 1 ac. ∴	17	13	3	= „ 2 ro.
1 ro. = $\frac{1}{2}$ of 2 ro. ∴	8	16	7.5	= „ 1 ro.
10 po. = $\frac{1}{4}$ of 1 ro. ∴	2	4	1.875	= „ 10 po.
8 po. = $\frac{1}{5}$ of 1 ro. ∴	1	15	3.9	= „ 8 po.
∴ by add^{n}	489	13	10.275	= „ 13 ac. 3 ro. 18 po.

Example 3. Find the value of 2 lb. 6 oz. 6 dwt. 14 gr. of gold at £4 4s. 11½d. per oz.

	£	s.	d.	
	4	4	11.5	= value of 1 oz.
∴ mult^{g}. by			30	(2 lb. 6 oz. being 30 oz.)
	127	8	9	= value of 2 lb. 6 oz.
5 dwt. = $\frac{1}{4}$ of 1 oz. ∴	1	1	2.875	= „ 5 dwt.
1 dwt. = $\frac{1}{5}$ of 5 dwt. ∴	0	4	2.975	= „ 1 dwt.
12 gr. = $\frac{1}{2}$ of 1 dwt. ∴	0	2	1.487...	= „ 12 gr.
2 gr. = $\frac{1}{6}$ of 12 gr. ∴	0	0	4.247...	= „ 2 gr.
∴ by add^{n}	128	16	8.58....	= „ 2 lb. 6 oz. 6 dwt. 14 gr.

Exercises. Set LXXXVI.

Find the cost of—

1. 3 ro. 20 po. at £16 18s. 4d. per ac.
2. 11 ac. 2 ro. 20 po. at £20 6s. 8d. per ac.
3. 9 ac. 1 ro. 25 po. at £1 12s. 6d. per ac.
4. 6 ac. 35 po. at £2 8s. 4d. per ro.
5. 1 qr. 14 lb. at £1 2s. 6d. per cwt.
6. 27 cwt. 3 qr. 7 lb. at £3 1s. 8d. per cwt.
7. 14 cwt. 2 qr. 21 lb. at £2 16s. 8d. per cwt.
8. 10 tons 6 cwt. 2 qr. at £4 14s. 6d. per ton.
9. 3 tons 19 cwt. 1 qr. at £2 10s. 4d. per ton.
10. 12 cwt. 3 qr. at £2 10s. 6d. per ton.
11. 5 tons 15 cwt. 1 qr. at £1 3s. 4d. per cwt.
12. 4 oz. 15 dwt. 8 gr. at 5s. 8d. per oz.
13. 1 oz. 13 dwt. 6 gr. at £4 2s. 6d. per oz.
14. 3 lb. 4 oz. 16 dwt. at £3 17s 10½d. per oz.
15. 5 yd. 1 ft. 8 in. at 2s. 10½d. per yd.
16. 20 sq. yd. 4 sq. ft. 30 sq. in. at 12s. 4d. per sq. yd.
17. 1 sq. yd. 6 sq. ft. 12 sq. in. at 4s. 8d. per sq. yd.
18. 3 qr. 5 bus. 2 pk. at £1 12s. 6d. per qr.
19. 1 qr. 6 bus. 1 pk. at £1 11s. 4d. per qr.
20. 3 tons 17 cwt. 3 qr. at £6 16s. 8½d. per ton.
21. 4 cwt. 2 qr. 23 lb. at £1 16s. 4½d. per cwt.
22. 2 tons 5 cwt. 24½ lb. at £1 10s. 8d. per cwt.
23. 2 ac. 3 ro. 32½ sq. po. at £12 6s. 6d. per ac.
24. 5 ac. 1 ro. 37 sq. po. at £10 4s. 8d. per ac.
25. 16 cwt. 24 lb. 12 oz. at £10 12s. per ton.
26. 5 cwt. 12 lb. 10 oz. at £15 6s. 8d. per ton.
27. 16 ac. 36 sq. po. 22 sq. yd. at £2 16s. 10d. per ro.
28. 5 ac. 15 sq. po. 11 sq. yd. at £6 10s. 9d. per ac.
29. 1 ac. 5 sq. po. 5½ sq. yd. at £7 3s. 6d. per ro.
30. 6 fur. 25 po. 3 yd. at £35 per mile.
31. 4 fur. 16 po. 2½ yd. at £5 6s. 8d. per fur.
32. 30 po. 4 yd. at £20 10s. 8d. per mile.

144. The following exercises will serve for the double purpose of revising the various cases above referred to and of familiarising the learner with the *rendering of an account*, or the *making out of a bill or invoice* by a tradesman to his customer.

Example. James Beck buys from his grocers, Ridgway

and Co., 10, Upton Street, Manchester, on January 4, 1875, 23 lb. of sugar at 4½d. per lb., and a cheese weighing 17 lb. 3 oz. at 8d. per lb.; on January 11, 5 doz. candles at 3s. 4d. per doz.; and on Feb. 19, a chest of tea containing 56¼ lb. at 2s. 4d. per lb. A bill for these goods is to be made out on March 31. How would this be done?

10, Upton Street, Manchester,
31*st March*, 1875.

Mr. James Beck
Bought of Ridgway and Co., Grocers, &c.

1875.					
January 4	23 lb. sugar	@ 4½d.		8	7½
„ „	17 lb. 3 oz. cheese	@ 8d.		11	5½
„ 11	5 doz. candles	@ 3s. 4d.		16	8
February 19	56¼ lb. tea	@ 2s. 4d.	6	11	3
		£	8	8	

Exercises. Set LXXXVII.

1. Jones and Williams, grocers, Edinburgh, forward to Mrs. Watson on 29th March, 1875, 16 lb. of tea at 3s. 4d. per lb., 8 lb. of tea at 4s. 2d. per lb., 3 lb. of butter at 1s. 4½d. per lb., 3 doz. of eggs at 9½d. per doz., and 7 lb. of biscuits at 5½d. per lb. Make a copy of the bill which is sent with the goods.

2. R. W. Smith, grain merchant, Leith, forwarded to Hugh Brown on 23rd June, 1875, 11 bus. of peas at 2s. 7d. per bus., 8 qr. of oats at 29s. per qr., 7 qr. of rye at £1 1s. 8d. per qr., 5 qr. of wheat at 43s. 5d. per qr., and 13 bus. of beans at 3s. 6d. per bus. Make a copy of the invoice accompanying the goods.

3. James Thomson bought from Watson and Co., wine merchants, Hill Street, Liverpool, on 1st February, 1875, 12 doz. bottles of port wine at 62s. per doz., and 18 doz. of claret at 44s. 6d. per doz.; on 3rd March 30 bottles of brandy at 3s. 4d. per bottle, and 15 gall. of gin at 15s. 3d. per gall.; and on 10th March 14 doz. of hock at 69s. 6d. per doz. Make a copy of a bill for these goods rendered on 15th of May.

4. Weir and Co., ironmongers, Glasgow, supplied James Nisbet in 1875 with the following articles, for which they send in an account at the end of the year:—On 10th May 1 doz. knives at 7½d. each, 1 doz. gimlets at 5½d. each, 2 doz. corkscrews at 4¾d. each; on 4th August 1 doz. knives at 1s. 6½d. each, 3 doz. files at 1s. 0½d. each; on 1st November 1½ doz. coffee-pots at 2s. 3d. each. Make a copy of the account.

5. James Mason, Regent Street, London, sends to Campbell and Co. on 5th March, 1875, 1 doz. linen shirts at 7s. 6d. each, 10 doz. linen handkerchiefs at 2s. 3d. each, 5 doz. silk handkerchiefs at 4s. 4½d. each, 6 doz. pairs of kid gloves at 3s. 2d. per pair, 4 doz. pairs of cotton gloves at 7½d. per pair, 12 doz. linen collars at 6s. 6d. per doz., 5½ doz. scarfs at 2s. 3½d. each, and 1½ doz. pairs of socks at 1s. 8d. per pair. Make a copy of the accompanying invoice.

6. William Paton receives from Irvine and Ure, stationers, Preston, for ready money, on 8th July, 1875, 12 quires of note-paper at 6s. 3d. per ream, 550 envelopes at 8s. 4d. per 1000, 225 quills at 2s. 6d. per hundred, 7 doz. steel pens at 4s. 6d. per gross, and 52 school slates at 3s. 3d. per doz. Make a copy of the receipted account.

7. Mrs. Wilson receives from Thomson and Yuill, stationers, Glasgow, on 8th January, 1875, 150 envelopes at 6s. per 1000, 250 envelopes at 4s. 6d. per 1000, 6 doz. penholders at ½d. each, 4 doz. copy-books at 2¾d. each, 8 doz. pens at 1½d. per doz., and half a ream of note-paper at 3½d. per quire, for which an account is rendered next day. Make a copy of this account.

8. James and Dowell, upholsterers, Glasgow, supply to Eustace Mansfield, on 17th February, 1875, 80 yd. of Brussels carpet at 5s. 9d. per yd., 73 yd. of drugget at 2s. 6d. per yd., 52 yd. of stair carpet at 3s. 9d. per yd., 4 doz. stair-rods at 7s. 6d. per doz., 8 doz. stair-rod eyes at 10½d. per doz., and charge 4d. per yd. for making and laying. Make a copy of the bill sent in on 24th February.

9. James Wilson, draper, receives from Herbert and Co., Manchester, on 5th January, 1875, 123 yd. of calico at 7¾d. per yd., 568 yd. of linen at 1s. 1d. per yd., 304 yd. of black silk at 8s. 9d. per yd., 186 yd. of coloured silk at 4s. 8d. per yd., 274 yd. of superfine broad cloth at 18s. 10d. per yd. Make a copy of the accompanying invoice.

10. William Strong receives from Ashton and Co., Edgeroad Colliery, on 19th May, 1875, 256 tons of coal at 18s. 6d. per ton, and 124 tons dross at 4s. 6d. per ton; on 26th May 302 tons of coal at £1 0s. 6d. per ton; on 3rd June 112 tons of coal at 19s. 9d. per ton. Make a copy of the bill rendered on 31st July.

11. James Thorn buys from Willis and Co., Sunderland, on 5th May, 1875, ¼ yd. of velvet at 15s. 6d. per yd., ⅞ yd. of woollen repp at 4s. 4d. per yd., 8¾ yd. of silk at 12s. 10d. per yd., 13½ yd. of ribbon at 1s. 6½d. per yd., and 31⅜ yd. of braid at 1½d. per yd. Make a copy of the bill sent next day.

12. Mrs. Pierce buys for ready money from Wenham and Dale, Dundee, on 1st October, 1875, the following articles:—2¼ lb. of butter at 1s. 10d. per lb., 3½ lb. of tea at 3s. 4½d. per lb., ¾ lb. of mustard at 1s. 5d. per lb., 5¾ lb. of sugar at 5d. per lb., 15¼ lb. of sugar at 6d. per lb., and 3 boxes of starch, each containing 2⅞ lb. at 4d. per lb. Make a copy of the receipted account.

13. William Robinson receives from Watkins and Co., Bolton, on 9th April, 1875, 204½ yd. of silk at 4s. 8½d. per yd., and 165⅝ yd. of ribbon at 1s. 4d. per yd.; on 18th August 361¾ yd. of satin at 8s. 10d. per yd., and 112¼ yd. of brocade at 18s. 4d. per yd. Make a copy of the account rendered at the latter date.

14. James Hilton, Shoreham, supplied Mrs. Whitefield on 24th December, 1874, with 15 lb. 10 oz. of beef at 8d. per lb., and 3lb. 8 oz. of suet at 7½d. per lb.; and on 1st January, 1875, 3 lb. 2 oz. of chops at 1s. 0½d. per lb., 10 lb. 5 oz. of mutton at 10d. per lb., and 2 lb. 3 oz. of suet at 8d. per lb. Make a copy of the bill sent in at the latter date.

15. John Morton, grain merchant, Hexham, supplies David Jones on 15th July, 1875, with 3 qr. 5 bus. of wheat at 50s. 4d. per qr., 2 qr. 2 bus. 1 pk. of oats at 34s. 8d. per qr., and 5 bus. 1 pk. of beans at 6s. 2d. per bus.; and on 12th August 115 qr. 3 bus. of wheat at 52s. 6d. per qr., and 6 bus. 1 pk. of barley at 37s. 8d. per qr. Make a copy of the account rendered on 26th August.

16. William Hobson buys of Walter Stubbs, Wastefield, on 15th July, 1875, 12 ac. 3 ro. 20 po. of potatoes at £25 12s. 6d. per ac.; on 21st July 6 ac. 30 po. of potatoes at £28 18s. 6d. per ac.; on 11th August 2 ro. 15 po. of turnips at £20 16s. 8d. per ac.; on 13th August 5 ac. 1 ro. 6 po. of turnips at £21 10s. 9d. per ac.; and 1 ac. 2 ro. 25 po. of beet-roots at £22 8s. per ac. Make a copy of the account as rendered and paid on 30th September.

17. John Docherty receives from Miller and Co., Waterside, on 23rd April, 1875, 16 tons 5 cwt. 1 qr. of household coal at 17s. 6d. per ton, and 25 tons 15 cwt. of steam coal at 11s. 8d. per ton; on 30th April 12 tons 13 cwt. 2 qr. of household coal at 18s. 9d. per ton, and 101 tons 3 cwt. of steam coal at 12s. 3d. per ton; and on 1st May 21 tons 1 cwt. 1 qr. of steam coal at 12s. 6d. per ton. Make a copy of the bill rendered on 31st May.

18. W. and J. Miller buy from Westdale and Co., importers, London, on 19th March, 1875, 65.6 metres of silk at 12s. 6d. per metre; on 17th June 3 doz. loaves of French sugar, each weighing 3.45 kilograms, at 9¾d. per kilogram, and 2 doz. loaves, each weighing 4.75 kilograms, at 10d. per kilogram; and on 15th July 2.64 kilograms of insulated copper wire (No. 30) at £1 2s. 6d. per kilogram, and 15.23 kilograms of insulated copper wire (No. 16) at 12s. 8d. per kilogram. Make a copy of the bill sent in with the goods.

145. We have seen that if the price of one unit be given, the price of any quantity expressed in terms of that unit is found by multiplying the given price by the number used in specifying the said quantity. Thus, knowing that the price of 1 yd. of cloth is 4s. 8d., we find the price of 5⅝ yd. by multiplying 4s. 8d. by 5⅝.

It, therefore, evidently follows that, conversely, if the price of any quantity expressed in terms of a single unit be given, the price of *one* such unit will be found by *dividing* the given price by the number used in specifying the corre-

sponding quantity. Thus, knowing the price of $5\frac{3}{8}$ yd. of cloth to be £1 5s. 1d., we obtain the price of 1 yd. by dividing £1 5s. 1d. by $5\frac{3}{8}$.

Example 1. If the cost of $\frac{4}{7}$ cwt. of coal be 1s. 2d., what is the cost of 1 cwt.?

$$\begin{aligned} &\text{The cost of } \tfrac{4}{7} \text{ cwt.} = 1\text{s. } 2\text{d.} \\ \therefore \quad &\quad ,, \qquad 1 \text{ cwt.} = 1\text{s. } 2\text{d.} \div \tfrac{4}{7} \\ &\qquad\qquad\qquad = 1\text{s. } 2\text{d.} \times \tfrac{7}{4} \\ &\qquad\qquad\qquad = 2\text{s. } 0\tfrac{1}{2}\text{d.} \end{aligned}$$

Example 2. If the price of 2.35 tons of iron be £9 15s. 10d., what will 1 ton cost?

$$\begin{aligned} &\text{The cost of } 2.35 \text{ tons} = £9\ 15\text{s. } 10\text{d.} \\ \therefore \quad &\quad ,, \qquad 1 \text{ ton} = £9\ 15\text{s. } 10\text{d.} \div 2.35 \\ &\qquad\qquad\qquad = £9\ 15\text{s. } 10\text{d.} \div 2\tfrac{7}{20} \\ &\qquad\qquad\qquad = £9\ 15\text{s. } 10\text{d.} \times \tfrac{20}{47} \\ &\qquad\qquad\qquad = £4\ 3\text{s. } 4\text{d.} \end{aligned}$$

The operation of division in these examples may also be justified without any reference to the preceding paragraph. Thus in the case of Example 1 we might reason as follows:—

$$\begin{aligned} &\text{The cost of four-sevenths of a cwt.} = 1\text{s. } 2\text{d.} \\ \therefore \quad &\quad ,, \quad \text{one-seventh} \quad ,, \quad = 1\text{s. } 2\text{d.} \div 4 \\ \text{and } \therefore \quad &\quad ,, \quad \text{a whole cwt.} \qquad = (1\text{s.}2\text{d.} \div 4) \times 7 \end{aligned}$$

the result being thus clearly obtained by operations equivalent to *dividing by* $\frac{4}{7}$.

Exercises. Set LXXXVIII.

1. If the cost of $\frac{1}{2}$ yd. of cloth be 3s. $7\frac{1}{4}$d., what will 1 yd. cost?
2. The cost of $\frac{3}{4}$ yd. of cloth being 5s. $9\frac{3}{4}$d., what is the price per yd.?
3. The price of $\frac{1}{10}$ ton of coal is 1s. $5\frac{1}{2}$d. At this rate what will 1 ton cost?
4. For $\frac{7}{10}$ ton of coal 11s. 8d. is paid. What is the price of it per ton?
5. What will 1 yd. of cloth cost at the rate of 7s. 6d. for $\frac{5}{8}$ yd.?
6. What will 1 cwt. of sugar cost, if the cost of $\frac{7}{12}$ cwt. be £1 8s. $3\frac{1}{2}$d.?
7. At the rate of 22s. 6d. for $\frac{3}{10}$ ton of iron, what should 1 ton cost?

8. When the cost of $\frac{13}{27}$ sq. yd. of ground is £1 8s. 6d., what ought 1 sq. yd. to cost?

9. The cost of $2\frac{3}{8}$ ton of iron is £10 18s. 6d. What is the cost per ton?

10. $12\frac{1}{3}$ oz. of silver are worth £3 6s. $6\frac{7}{12}$d. What is 1 oz. worth?

11. What must I pay for 1 lb. of tea, if the rate be £1 4s. $4\frac{1}{2}$d. for $6\frac{1}{2}$ lb.?

12. $12\frac{3}{8}$ yd. of velvet are sold for £9 14s. 3d. What is the selling price per yd.?

13. 1 sq. po. of ground is sold for £6 11s. 1d. What is the cost of it per sq. yd.?

14. 10 bars of copper, each weighing $2\frac{1}{3}$ cwt., are worth £111 8s. 4d. What is the value per cwt.?

15. The charge for 12.15 metres of wire is 18s. 3d. What would a metre of it cost?

16. What is the cost per ton of iron when .0125 ton is sold for 1s. 1d.?

17. $6\frac{3}{4}$ yd. of one kind of silk are bought for £4 4s. $4\frac{1}{2}$d., and $10\frac{3}{8}$ yd. of another kind for £6 14s. 7d. Which is the dearer, and by how much per yd.?

18. If $\frac{3}{4}$ of $\frac{5}{7}$ of a ship be worth £895 2s. 6d., what is the whole worth?

19. 5 metres of cloth are bought for £3 16s. 8d. Find the cost per yard, 1 metre being taken equal to 1.0936 yd.

20. For 1 oz. Troy or 31.103 grams of gold £4 4s. 10d. is received. Find the value of 1 gram of gold at this rate?

146. In the examples under § 143 of finding the cost of a quantity of goods, the rate is specified by giving the price *per yard*, *per ton*, &c.; that is to say, always the price of *one* unit. It is well known already, however, that this is not the only way of specifying such a rate. Instead of being told that the price of cloth is so much per one yard, we may only know that it is at the rate of so much for, say, $7\frac{1}{4}$ yd.; or we may be asked to calculate the value of a field containing a certain number of acres from knowing the value of, say, 5.375 ac., instead of the value per acre. But then, having learned in the preceding paragraph how to find the cost of 1 yd. from the cost of $7\frac{1}{4}$ yd., or the value of 1 acre from that of 5.375 ac., the mode of procedure in such cases will be almost at once manifest.

Example 1. The cost of $3\frac{1}{5}$ ac. of land is £16 2s. 8d. What would .075 ac. cost at this rate?

$$
\begin{array}{llll}
 & \text{The cost of} & 3\tfrac{1}{5}\text{ ac.} & = £16\ 2s.\ 8d. \\
\therefore & \quad ,, & 1\text{ ac.} & = £16\ 2s.\ 8d. \div 3\tfrac{1}{5} \\
\text{and } \therefore & \quad ,, & .075\text{ ac.} & = (£16\ 2s.\ 8d. \div 3\tfrac{1}{5}) \times .075 \\
 & & & = (£16\ 2s.\ 8d. \times .075) \div 3\tfrac{1}{5} \\
 & & & = £16\ 2s.\ 8d. \times \tfrac{3}{40} \times \tfrac{5}{16} \\
 & & & = £16\ 2s.\ 8d. \times \tfrac{3}{128} \\
 & & & = £48\ 8s. \div 128 \\
 & & & = 7s.\ 6\tfrac{3}{4}d.
\end{array}
$$

Example 2. For 5 guineas I can buy $3\frac{3}{14}$ cwt. of sugar. Find the cost at this rate of $5\frac{1}{7}$ lb.

$$
\begin{array}{llll}
 & \text{The cost of} & 3\tfrac{3}{14}\text{ cwt.} & = 105s. \\
\therefore & \quad ,, & 1\text{ lb.} & = 105s. \div (3\tfrac{3}{14} \times 112) \\
\text{and } \therefore & \quad ,, & 5\tfrac{1}{7}\text{ lb.} & = 105s. \times \dfrac{5\frac{1}{7}}{3\frac{3}{14} \times 112} \\
 & & & = 105s. \times \dfrac{5\frac{1}{7}}{360} \\
 & & & = 105s. \times \tfrac{1}{70} \\
 & & & = 1s.\ 6d.
\end{array}
$$

After a little practice the learner may dispense with the first two lines of this process, finding himself able to write down the third at once with correctness.

Example 3. What is the value of $\frac{1}{7}$ of $\frac{3}{4}$ of a ship when $\frac{1}{8}$ of $\frac{2}{3}$ of it is valued at 500 guineas?

$$
\begin{array}{ll}
\text{The value of } \tfrac{1}{7} \text{ of } \tfrac{3}{4} \text{ of the ship} & = £525 \div (\tfrac{1}{8} \times \tfrac{2}{3}) \times (\tfrac{1}{7} \times \tfrac{3}{4}) \\
 & = £525 \times 12 \times \tfrac{3}{28} \\
 & = £75 \times 12 \times \tfrac{3}{4} \\
 & = £75 \times 9 \\
 & = £675.
\end{array}
$$

Exercises. Set LXXXIX.

1. Find the cost of $3\frac{3}{4}$ yd. at the rate of £1 5s. for $5\frac{3}{8}$ yd.

2. $7\frac{1}{12}$ railway shares are worth £673 10s. What is the value of $3\frac{3}{5}$ shares at this rate?

3. Find the cost of 1.2 cwt. at the rate of 12s. 4d. for $\frac{3}{8}$ cwt.

4. 2.12 kilog. of raw sugar are sold for 1.59 fr. What would 11.9 kilog. cost at this rate?

5. At the rate of £11 2s. 8d. for $17\frac{3}{8}$ yd. of cloth, what would be the cost of 5 pieces each containing $49\frac{1}{4}$ yd.?

6. The value of $\frac{1}{4}$ oz. of gold is £1 1s. 3d. Find the value at this rate of the third part of a nugget of gold weighing $3\frac{1}{4}$ oz.

7. The cost of $\frac{3}{4}$ of $\frac{7}{8}$ of a ship is £104 12s. 6d. What is the value of $\frac{3}{14}$ of it?

8. What would an ounce and a half of tea cost at the rate of a crown for a pound and a quarter?

9. If the cost of $3\frac{1}{2}$ lb. of beef be 2s. 11d., what will $\frac{3}{8}$ cwt. cost?

10. What is the cost of $2\frac{3}{4}$ bushels of wheat at the rate of $5\frac{1}{2}$ qr. for £14 11s. 6d.?

11. If £$5\frac{3}{11}$ be the value of 2.9 oz. Troy of an alloy of gold, what would $2\frac{3}{8}$ lb. be worth at the same rate?

12. Calculate the cost of 1 cwt. 1 lb. $1\frac{3}{8}$ oz. at the rate of £1 4s. 6d. for $1\frac{3}{4}$ st.

13. What are $55\frac{1}{2}$ sovereigns worth in French money at the rate of 18.75 fr. for 15s.?

14. A piece of land containing 5.85 ac. costs £231 6s. $4\frac{1}{2}$d. What would a piece $3\frac{1}{8}$ ro. larger cost at this rate?

15. A milkman buys $76\frac{3}{4}$ pints of milk for £1 5s. 7d., adds $7\frac{1}{8}$ pints of water, and sells the mixture at the rate paid for the milk. What would he receive?

16. A certain cloth is sold at the rate of $3\frac{1}{8}$ yd. for 12s. 6d.; $173\frac{1}{8}$ yd. of it are bought to make 13 dresses of equal size. What would the cloth of 1 dress cost?

17. For 1 oz. Troy of silver 5s. $11\frac{18}{37}$d. is received at the Mint. What would 4.44 lb. avoird. of silver be worth at this rate?

18. A field containing $12\frac{3}{8}$ ac. and worth £254 2s., is divided into 15 equal plots after $6050\frac{1}{4}$ sq. yd. have been set off as a common. Find the value of one of the plots.

19. Find the price of $\frac{3}{8}$ of a cask of brandy containing 1.56 hectolitre at the rate of 22s. 6d. per gallon.

20. A wire $2\frac{1}{4}$ mi. long is bought for £6 12s., and weighs .048 oz. per fathom. Find the cost of 2.35 lb. of it?

147. The foregoing examples will have been recognised to be of the same kind as those considered in §§ 64—67, and headed "Practical Examples involving both Multiplication and Division." They all belong, however, to a single class of those examples, viz., those in which the subject is the cost of goods, &c., and have again been brought forward in order to show that when the numbers involved are fractional the mode of procedure is unaltered; in other words, to prove that the processes of multiplication

and division are as valid in the case where the numbers involved are fractions as in the case where they are integers.

We have now further to observe that not only is this true of the single class referred to, but that it might have been as easily established in regard to any of the others. In all of them there is *a calculation to be made from a specified rate;* and this characteristic, which is the origin of the mode of procedure, ought never to be lost sight of.

Example 1. In $9\frac{1}{2}$ hours a band of reapers can reap $53\frac{1}{8}$ ac. At this rate how long would they take to reap 371 ac. $3\frac{1}{2}$ ro.?

$$53\tfrac{1}{8}\text{ ac.} = 212\tfrac{1}{2}\text{ ro.}$$
$$\text{and}\quad 371\text{ ac. }3\tfrac{1}{2}\text{ ro.} = 1487\tfrac{1}{2}\text{ ro.}$$

Now to reap $212\frac{1}{2}$ ro. they take $9\frac{1}{2}$ hr.

$\therefore$ " 1 ro. " $9\frac{1}{2}$ hr. $\div 212\frac{1}{2}$

and $\therefore$ " $1487\frac{1}{2}$ ro. " $(9\frac{1}{2}\text{ hr.} \div 212\frac{1}{2}) \times 1487\frac{1}{2}$

i.e., $\frac{19}{2}\text{ hr.} \times \frac{2}{425} \times \frac{2975}{2}$

i.e., $\frac{19}{2}\text{ hr.} \times \frac{1}{1} \times \frac{7}{1}$

i.e., $66\frac{1}{2}$ hr.

Example 2. For a certain sum of money a package weighing 3.75 cwt. is carried $2\frac{1}{5}$ mi. At this rate how far ought a package weighing $6\frac{1}{2}$ st. to be carried for the same sum?

$$3.75\text{ cwt.} = 8\text{ st.} \times 3.75 = 30\text{ st.}$$

Now 30 st. are carried $2\frac{1}{5}$ mi.

$\therefore$ 1 st. ought to be carried $2\frac{1}{5}\text{ mi.} \times 30$

and $\therefore$ $6\frac{1}{2}$ st. " " $(2\frac{1}{5}\text{ mi.} \times 30) \div 6\frac{1}{2}$

which $= \frac{11}{5}\text{ mi.} \times 30 \times \frac{2}{13}$

$= \frac{132}{13}$ mi.

$= 10\frac{2}{13}$ mi.

The learner is already familiar with the fact that in some of the examples of this kind (for instance, the former of the two preceding) the first operation necessary is *division;* while in others (for instance, Example 2 above) it is *multiplication* that is first needed. As a consequence of this, if it be desired, for shortness' sake, to omit as before the first two lines of the process, some little care is necessary to insure that the two numbers have had assigned them their proper

operations—that what in correctness should be the multiplier has not been made the divisor, and *vice versâ*. For this purpose the learner may either simply *reason* as before, omitting to *write* until the final step is made; or, what is perhaps easier, ascertain whether the required result is to be greater or less than the corresponding quantity mentioned in the specification of the rate, and take for multiplier the greater or less of the two numbers accordingly. The following illustrations will serve to make this fully understood.

Example 3. A railway engine runs $75\frac{1}{2}$ mi. in $2\frac{1}{4}$ hr. At this rate how far would it go in 13 min. 20 sec.?

$$\begin{aligned} 13 \text{ min. } 20 \text{ sec.} &= 800 \text{ sec.} \\ \text{and } 2\tfrac{1}{4} \text{ hr.} &= 8100 \text{ sec.} \\ \therefore \text{distance required} &= 75\tfrac{1}{2} \text{ mi.} \times \frac{800}{8100} \\ &= \frac{151}{2} \text{mi.} \times \frac{8}{81} \\ &= \frac{604}{81} \text{ mi.} \\ &= 7\tfrac{37}{81} \text{ mi.} \end{aligned}$$

Here, on reading the question, we (1) at once prepare for giving the answer by writing down "Distance required ="; (2) we ascertain what distance is mentioned in specifying the rate, write this distance after the sign of equality, and follow it with the sign of multiplication; (3) we ask ourselves "If it go $75\frac{1}{2}$ miles in 8100 seconds, will it go a greater or less distance in 800 seconds?" and seeing at once that the distance will be less, and that, therefore, the less of the two numbers must be taken for multiplier, we write $\frac{800}{8100}$ after the sign of multiplication.

Example 4. A quantity of wheat is cut in $7\frac{1}{3}$ hr. by 16 men. Supposing the men to share the work equally, how long at this rate would 11 of them have taken?

$$\begin{aligned} \text{Time required} &= 7\tfrac{1}{3} \text{ hr.} \times \frac{16}{11} \\ &= \frac{22}{3} \text{ hr.} \times \frac{16}{11} \\ &= \frac{2}{3} \text{ hr.} \times \frac{16}{1} \\ &= 10\tfrac{2}{3} \text{ hr.} \end{aligned}$$

Here, as before, (1) on seeing the question "How long," &c., we at once write "Time required="; (2) on observing the time mentioned in specifying the rate we follow this up with "$7\frac{1}{8}$ hr. ×"; and (3) having got the answer to the question, "If 16 men take $7\frac{1}{8}$ hr., will 11 men take more or less?" we complete the line with "$\frac{16}{11}$."

Exercises. Set XC.

1. A ship sails $154\frac{11}{18}$ mi. in 18 hr. At this rate how far would it sail in $4\frac{1}{2}$ hr.?

2. $9\frac{15}{16}$ tons of coal are consumed by an engine in $5\frac{5}{8}$ da. At this rate how much will be used in $29\frac{7}{10}$ da.?

3. A greyhound makes 163 leaps in two-thirds of a minute. How many will it make at this rate in seven minutes and a quarter?

4. A fountain spouts forth 125.625 gall. of water in an hour and a quarter. How much water would be used if the fountain played constantly at this rate for 3 da. $2\frac{1}{8}$ hr.?

5. To sow a field of 23.78 ares 3.5 decalitres of wheat are necessary. At this rate how much would be required to sow another field 54.12 ares in extent?

6. A workman receives 39.05 francs for 88 hours' work. For how many hours at this rate will his wage be 14.2 francs?

7. A pump raises $18\frac{1}{2}$ cub. ft. of water in 14 minutes. At this rate how much water would be raised in $1\frac{1}{4}$ hr.?

8. How long must a pump be kept going at the rate mentioned in the preceding exercise to raise $52\frac{4}{5}$ cub. yd. of water?

9. $8\frac{3}{4}$ yd. of cloth cost £1.725. At this rate how much may be bought for £2.76?

10. $6\frac{1}{8}$ bus. of wheat are sown, the produce of which in the autumn is $73\frac{3}{4}$ bus. At this rate how much must be sown to yield 590 bus.?

11. A foot and a half of gilt wire weighs $\frac{1}{4000}$ oz. Calculate at this rate the length of 3.775 lb. of it.

12. What quantity of carbon will be produced from $1\frac{1}{2}$ cwt. of wood, supposing 10 lb. of wood give $2\frac{1}{8}$ lb. of carbon, and the calculation be made at this rate?

13. The magnitude of a certain angle is $2\frac{1}{8}$ radials. Knowing that $180° = 3.14159\ldots$ radials, express the magnitude of the angle in degrees, &c.

14. What quantity of wine can be bought for 17.35 francs at the rate of a franc and a half per bottle containing .85 litre?

15. A vertical rod 1 metre long throws a horizontal shadow $4\frac{1}{2}$ feet in length. What must be the length of a chimney which at the same time throws a horizontal shadow $58\frac{1}{8}$ yd. long?

16. Express the magnitude of an angle of $6° \, 14' \, 22\frac{1}{2}''$ in terms of the radial. (See Ex. 13.)

17. If in $3\frac{5}{8}$ da. $10\frac{11}{21}$ tons of coal be used for an engine, how long

at the same rate would 17 waggons of coal last, each containing $3\frac{3}{7}$ tons?

18. How long would a ship take, at the rate mentioned in Exercise 1, to sail 15 knots?

19. For 51.75 Austrian florins 105.3 francs are obtained. At this rate what is the equivalent in Austrian money of 11.7 francs, and what, in French money, of 17.71 florins?

20. A merchant pays weekly for flour £5 11s. 3d. when it is selling at £$1\frac{7}{8}$ per cwt. What ought he to pay when flour is half-a-crown per cwt. dearer, supposing he continues to receive the same quantity?

21. A new sovereign weighs $\frac{40}{1869}$ lb. Troy, and loses after a time by usage 2.304 grains of its weight. What ought it then to be worth?

22. A boat's crew of five have provisions for a month, the allowance of flour being $2\frac{3}{5}$ lb. per man per day. How much must this allowance be reduced so as to supply a crew of seven for the same time?

23. A traveller walking $12\frac{3}{7}$ hr. per day finishes half of his journey in $8\frac{3}{4}$ da. How long would he take to finish the rest, walking at the same rate $10\frac{6}{13}$ hr. per day?

24. $25\frac{1}{2}$ lb. of flour are used in making $36\frac{1}{4}$ lb. of cake. What weight of other ingredients besides flour must there be in 10 lb. of this cake?

25. A grocer sells tea at the rate of $3\frac{1}{4}$ lb. for half a guinea. Supposing the cost price of this quantity was 8s. 4d., what profit would he make in selling $115\frac{1}{4}$ lb.?

26. A printer undertakes to fold the sheets of a book in $3\frac{1}{4}$ hr., and for this purpose finds it necessary to employ 67 girls. Before beginning, however, a delay of $2\frac{1}{3}$ hr. is granted him. Supposing all the girls to work throughout at the same rate, how many fewer will now suffice to finish the work in time?

27. The bulk of 6.061 oz. of pure silver is the same as the bulk of 11.145 oz. of pure gold. What weight of silver has the same bulk as 6687 grains of gold?

28. What must be the actual distance between two places, if the distance between them on a map drawn on a scale of 6 inches to a mile be $14\frac{3}{11}$ inches?

29. If a bar of iron at the temperature of freezing water (0°) measures 1 yd., it will measure 1.00118 yd. when at the temperature of boiling water (100°). Accepting this, calculate the length at 0° of a bar which at 100° is 4.356 yd. long.

30. 15.23 lb. of oil when used as engine fuel are found to have the same effect as 35 lb. of wood. At this rate how much oil would be required per year of 287 days to supply an engine which consumes $1\frac{3}{4}$ tons of wood per day?

31. A piece of rock weighing $3\frac{1}{2}$ lb. is put into a vessel already quite full of water, and it is found that 38.37 cub. in. of water overflow. What would be the weight of a cubic foot of this rock?

32. A steel foot-rule from expansion is found to be 12.15 inches in length, and in this state the length of a bar of iron is measured with

it, the apparent result being 16 ft. $3\frac{1}{2}$ in. Find, however, the true length.

33. Water weighs $\frac{5}{97}$ of the same bulk of gold, and a cubic foot of water weighs 62.5 lb. Calculate from these data the weight of 1.44 cub. in. of gold.

34. 100 lb. of corn are found to produce 76.6 lb. of meal and 21.3 lb. of bran, the remainder being waste. At this rate how much meal, bran, and waste would result from 1304 lb. of corn? And how much corn would be required to produce 1000 lb. of meal?

35. Three men reap a wheat field, and receive for their work 17s. 6d., £2 18s., £1 18s. 6d. respectively, being paid at the same rate per acre. Knowing that the first has cut $2\frac{1}{2}$ ac., find the size of the field.

36. A chronometer is set correctly at 1 p.m. on Monday, and is found to be 1.7 sec. behind at noon on the following Sunday. At this rate of going what would be the true time next day when it indicated 1 p.m.?

37. A dealer buys $\frac{9}{16}$ ton of potatoes for 42s. 8d. At what rate must he sell them per half-stone so as to gain 7s. 4d. on the transaction?

38. In the Réaumur thermometer the tubular space between the points marked "Freezing Point" and "Boiling Point" is divided into 80 equal parts; in the Centigrade thermometer the same space is divided into 100 equal parts; and the divisions in both are marked in order 1°, 2°, 3°, &c. Express 23.5 Réaumur degrees in Centigrade degrees, and 5.48 Centigrade degrees in Réaumur degrees.

39. In the Fahrenheit thermometer the space referred to in the preceding exercise is divided into 180 equal parts, and the divisions are numbered 33°, 34°, 35°, &c., "Freezing Point" being marked 32° instead of 0°, as in the other two instruments. What must correspond in this thermometer with 5°.6 Centigrade, and what in the Centigrade thermometer must correspond with 64°.2 Fahrenheit?

40. Complete the following table, which is meant to give six different readings of temperature expressed in each of the three ways above mentioned:—

	Réaumur.	Fahrenheit.	Centigrade.
1			18°.75
2			20°.20
3		72°.42	
4		76°.85	
5	33°.5		
6	42°.65		

148. The quantities whose rates of increase or decrease we have hitherto had examples of have been dependent

for their magnitude upon the magnitude of only *one* other quantity, *e.g.*, *distance* travelled upon *time* taken, *cost* of ribbon upon *length* purchased, &c. A quantity may, however, be dependent in this way upon *more than one* other quantity, and thus arise rates of less simple form. For example, a rate of so much *per ton per mile* may be charged for the conveyance of goods; and instead of the consumption of flour on board a ship being given as so much per day, it may be stated as so much *per man per day*. Here the cost of conveyance is dependent upon the *weight* of the goods and the *distance* they are carried; and the quantity of flour upon the *number of men* and the *time*.

Example 1. What quantity of tinned meat will supply a regiment of 670 men for 7 days at the rate of $7\frac{1}{2}$ oz. per man per day?

Quantity required by 1 man for 1 day $= 7\frac{1}{2}$ oz.
$\therefore$ „ 670 men „ $= 7\frac{1}{2}$ oz. $\times 670$
$\therefore$ „ 670 men for 7 days $= 7\frac{1}{2}$ oz. $\times 670 \times 7$
$= 35175$ oz.

Example 2. For the conveyance of $3\frac{1}{2}$ tons of luggage a distance of $30\frac{1}{4}$ miles the sum of 70s. 7d. is charged. At what rate is this per ton per mile?

The carriage of—
$3\frac{1}{2}$ tons over $30\frac{1}{4}$ mi. costs 70s. 7d.
$\therefore$ „ 1 ton „ $30\frac{1}{4}$ mi. „ 70s. 7d. $\div 3\frac{1}{2}$
$\therefore$ „ 1 ton „ 1 mi. „ (70s. 7d. $\div 3\frac{1}{2}) \div 30\frac{1}{4}$
i.e., 70s. 7d. $\times \frac{2}{7} \times \frac{4}{121}$
i.e., 70s. 7d. $\times \frac{8}{847}$
i.e., 8d.

Example 3. A contractor paid 27 workmen at the rate of $10\frac{1}{2}$d. per man per hour, and for a week's work the total sum they received was £59 1s. 3d. How many hours must they have worked?

Pay of 27 workmen for a week = £59 1s. 3d.
$\therefore$ „ 1 workman „ = £59 1s. 3d. $\div 27$
= £2 3s. 9d.

But pay of 1 workman for an hour $=10\frac{1}{2}$d.

$\therefore$ no. of working hours in a week $=$ £2 3s. 9d. $\div 10\frac{1}{2}$d.

$$=525\times\frac{2}{21}$$

$$=50.$$

Or thus :—

Pay of 1 workman per hour $=10\frac{1}{2}$d.

$\left.\begin{array}{llll}\therefore & \text{,, 27 workmen} & \text{,,} & =10\tfrac{1}{2}\text{d.}\times 27. \\ \text{But} & \text{,,} \quad\quad \text{,,} & \text{per week} & =\text{£}59 \text{ 1s. 3d.}\end{array}\right\}$

$\therefore$ no. of working hours in a week $=\dfrac{\text{£}59 \text{ 1s. 3d.}}{10\frac{1}{2}\text{d.}\times 27}$

$$=50.$$

Exercises. Set XCI.

1. One horse's food for one day costs 3s. $4\frac{1}{2}$d. At this rate what would be the cost of feeding 17 horses for a year?

2. What was the rate of expense per day for one person when a party of seven spent 47 guineas in 12 days?

3. The cost of keeping 13 horses for 15 weeks was found to be £190 10s. $7\frac{1}{2}$d. What was the cost per horse per day?

4. At the rate found in Exercise 2, what would be the expenses of a vacation party of five from 25th July to 13th September, both of these days being included?

5. What is the freight per cwt. per mile when a quantity of goods weighing 13 cwt. 1 qr. is carried $14\frac{1}{4}$ mi. for £1 12s. $0\frac{1}{4}$d.?

6. Six workmen build 4050 cub. ft. of wall in 7 days, working 8 hr. per day. Supposing they all work at the same rate, how much is built by one man in an hour?

7. At the rate found in Exercise 6, how many cubic feet of wall will 7 workmen build in a fortnight working 9 hours per day?

8. A money-lender charges £96 3s. 9d. for lending £320 12s. 6d. for 3 years. What is his rate of charge per pound per year?

9. A banker in return for having the use of money pays at the rate of £3 per £100 per year. What would he pay for the use of £680 15s. for $4\frac{1}{2}$ years?

10. A charitable institution pays at the rate of £9 15s. 6d. per boy per year for the maintenance of 150 boys. How much would the expenses be increased in 10 years by raising the number of boys to 185?

11. In a hall there are 4 gas chandeliers each having 85 jets, and the quantity of gas burned in $3\frac{3}{4}$ hr. is 3060 cub. ft., the price of which is £1 5s. 6d. Find (1) the amount of gas burned per jet per hour, (2) the cost per hour for each chandelier.

12. If the 150 boys in Exercise 10 could be maintained 7 years for £1000 5s., what would be the saving per boy per year?

13. The water made use of in 3 weeks by a town of 1020 inhabitants amounted to 428400 gall., but after improvements in taps, &c., it was found that the consumption was 158100 gall. in 10 days. How much less was this per inhabitant per day? and what would be the total quantity of water saved per year?

149. From these exercises we now pass to others bearing the same relation to them as those of §§ 146-7 bear to those of § 143; that is to say, we are now asked, for example, to calculate the travelling expenses of a party of 9 persons for 13 days, not from knowing the rate of expense per person per day, but from knowing that, say, £24 17s. was the outlay of 4 persons for 7 days. Of course it is at once seen that the expenses of 4 persons for 7 days being given, we are able to find, as in Set XCI., Ex. 2, the rate of expense per person per day, and thence, as in Set XCI., Ex. 4, can calculate the expenses required. But we may also reason as the following shows.

Example 1. The travelling expenses of 4 persons for 7 days amounted to £24 17s. Calculate at this rate the expenses of 9 persons for 13 days.

$$\begin{aligned}\text{Expenses required} &= £24\ 17\text{s.} \times \frac{9}{4} \times \frac{13}{7} \\ &= £24\ 17\text{s.} \times \frac{117}{28} \\ &= £103\ 16\text{s. } 9\text{d.}\end{aligned}$$

Here, neglecting at first the fact that the lengths of time mentioned are different, we proceed as if the exercise were, "The travelling expenses of 4 persons for seven days amounted to £24 17s. Calculate at this rate the expenses of 9 persons *for the same time.*" This the learner already knows how to perform, the necessary calculation being £24 17s. $\times \frac{9}{4}$. And now, knowing that the expenses of 9 persons for 7 days amount to £24 17s. $\times \frac{9}{4}$, we ask what they will be for 13 days, the answer to which we know in the same way to be

$$£24\ 17\text{s.} \times \frac{9}{4} \times \frac{13}{7}.$$

Example 2. How many hours ought 13 men to work for a total sum of £6 10s. 9¾d., if they be paid at the same rate as 11 others who receive £8 3s. 7½d. for 17 hours' work?

$$\begin{aligned}\text{Required no. of hours} &= 17 \times \frac{£6\ 10s.\ 9\tfrac{3}{4}d.}{£8\ 3s.\ 7\tfrac{1}{2}d.} \times \frac{11}{13}\\ &= 17 \times \frac{6279}{7854} \times \frac{11}{13}\\ &= 17 \times \frac{483}{714} \times \frac{1}{1}\\ &= \frac{483}{42}\\ &= 11\tfrac{1}{2}.\end{aligned}$$

Here we proceed at first as if the number of men were the same in the case regarding which inquiry is made as in the other case, viz., 11, and we ask ourselves the question, "How many hours ought 11 men to work for £6 10s. 9¾d. at the rate of £8 3s. 7½d. for 17 hours?" The result we easily see to be

$$17 \times \frac{£6\ 10s.\ 9\tfrac{3}{4}d.}{£8\ 3s.\ 7\tfrac{1}{2}d.};$$

and now, knowing that 11 men ought to work this time, we make inquiry how long 13 men ought to work instead, the answer to which we at once know will be obtained by multiplying by the less number and dividing by the greater, that is, by multiplying by $\frac{11}{13}$; so that the desired result in its unsimplified form is

$$17 \text{ hours} \times \frac{£6\ 10s.\ 9\tfrac{3}{4}d.}{£8\ 3s.\ 7\tfrac{1}{2}d.} \times \frac{11}{13}.$$

Instead, however, of beginning by considering the number of *men* to be the same in both cases, we might proceed at first as if the *sums paid* were the same. Doing this, we ask ourselves the question, "If 11 men receive £8 3s. 7½d. for 17 hours' work, how many hours ought 13 men to work at the same rate for the same sum?" the answer to which is

$$17 \text{ hours} \times \frac{11}{13}.$$

In the next place, 17 hr. × $\frac{11}{13}$ being thus found to be the time they ought to work for £8 3s. 7½d., we ask how long they

ought to work for £6 10s. $9\frac{3}{4}$d., the answer to which we see will be got by multiplying by $\frac{£6\ 10s.\ 9\frac{3}{4}d.}{£8\ 3s.\ 7\frac{1}{2}d.}$, the result in its unsimplified form being

$$17 \text{ hours} \times \frac{11}{13} \times \frac{£6\ 10s.\ 9\frac{3}{4}d.}{£8\ 3s.\ 7\frac{1}{2}d.},$$

which, of course, is essentially the same as before.

Example 3. Five brick-making machines turn out 825000 bricks in 110 hours. At this rate how long will three of them take to turn out 787500 bricks? Also, if they have 14 days to do it, how many hours per day must they work?

(1.)
$$\begin{aligned} \text{Time required} &= 110 \text{ hours} \times \frac{5}{3} \times \frac{787500}{825000} \\ &= 110 \text{ hr.} \times \frac{1}{1} \times \frac{2625}{1650} \\ &= \frac{2625}{15} \text{ hr.} \\ &= 175 \text{ hr.} \end{aligned}$$

The question here being "How long?" and the time specified in the given case being 110 hours, we at once write down, "Time required $=110$ hr. $\times$ "; then, thinking of the number of *machines* employed, we see that 3 will take longer than 5, and we therefore annex "$\frac{5}{3}$" to what was previously written; finally, referring to the number of *bricks*, we see that for the making of 787500 less time will be needed than for the making 825000, and we further annex " $\times\frac{787500}{825000}$."

(2.) Required number of hours per day $=\frac{175}{14}=12\frac{1}{2}$.

Example 4. In 11 days 5 brick-making machines, working 10 hours a day, turn out 825000 bricks. At this rate, how many hours a day ought 3 of them to work for 14 days in order to produce 787500 bricks?

A little consideration will show that the question here asked is exactly the same as the second of the two questions in Example 3, and the learner should satisfy himself of this before examining the following mode of procedure.

$$\text{Required number of hours per day} = 10 \times \frac{11}{14} \times \frac{5}{3} \times \frac{787500}{825000},$$
$$= 1 \times \frac{1}{14} \times \frac{5}{1} \times \frac{2625}{75},$$
$$= 1 \times \frac{1}{2} \times \frac{1}{1} \times \frac{375}{15}$$
$$= 12\tfrac{1}{2}.$$

Here we have proceeded as in the case of Examples 1 and 2, viz., in the first place, considering the number of machines and the number of bricks the same in both cases, the result being 10 hr. $\times \frac{11}{14}$; then, taking into consideration the fact that the number of machines is different, the result now being 10 hr. $\times \frac{11}{14} \times \frac{5}{3}$; and finally also the fact that the number of bricks is different, which leads us to annex another multiplier, $\frac{787500}{825000}$.

It will be found instructive, however, to proceed also in the way first alluded to, viz., by ascertaining as a starting-point the rate of brick-making per machine per hour:—

No. of bricks made by 5 machines in 110 hr. $= 82500$

$\therefore$ „ „ 1 machine in 1 hr. $= \frac{825000}{5 \times 110}$
$= 1500.$

$\therefore$ „ „ 3 machines in 1 hr. $= 4500$

But „ „ 3 machines in 14 da. $= 787500$

$\therefore$ number of working hours in 14 da. $= \frac{787500}{4500}$
$= 175$

$\therefore$ „ „ 1 da. $= 175 \div 14$
$= 12\tfrac{1}{2}.$

EXERCISES. SET XCII.

1. A travelling party of three persons spends £16 5s. 6d. in 11 days. At this rate what ought to be the expenses of a party of five travelling for 33 days?

2. The wages of 26 masons for 44 hours of work amount to £42 18s. At the same rate what would be the wages of 95 masons for 25 hours of work?

3. The cost of keeping 17 horses for 7 days is £11 18s. At the same rate what would it cost to keep 9 horses for 13 days?

4. What will be the wages of 3 workmen for 17½ hours of work, if the rate at which they are paid be the same as that which enables 13 men to receive £11 15s. 7½d. for 7½ hours?

5. In $7\frac{1}{2}$ hours 4 reaping machines cut 47 ac. 20 sq. po. of wheat. How much would be cut at this rate in 3 hr. 20 min. by 3 machines?

6. A quantity of luggage weighing $17\frac{1}{2}$ cwt. is carried 13 miles for £9 19s. $0\frac{3}{4}$d. What would be the cost at this rate of conveying $2\frac{1}{4}$ cwt. a distance of 113 miles?

7. For the loan of £100 for 2 years a money-lender receives £4. At this rate what ought he to receive for lending £540 for $3\frac{1}{2}$ years?

8. A money-lender lends £50 for 2 years and charges £3 10s. At this rate what would be his charge for lending £875 for 7 years?

9. A sum of £100 is lent for 365 days at a charge of £7 15s. What at this rate would be the charge for lending £1160 10s. for 511 days?

10. The sum of £6 10s. is given for the loan of £60 for $1\frac{1}{2}$ year. What should be given at this rate for the loan of £540 for $4\frac{1}{2}$ years?

11. The allowance of flour for a company of 75 soldiers for 22 days is $2062\frac{1}{2}$ lb. At this rate how much would be required for a company of 180 soldiers for 13 days?

12. Five water-pipes of one size allow 9600 gall. of water to pass through them into a reservoir in $2\frac{1}{2}$ hr. If one of them were closed, how much water at this rate would flow through the others in 4 hr.?

13. 2475 lb. of flour are required to provision a company of 150 soldiers for 11 days. At the same rate how long would 840 lb. serve a company of 35?

14. 12 bricklayers for a week's work of 50 hr. receive in all £22 10s. At the same rate how many hours should 7 of them work to receive in all £8 13s. 3d.?

15. For the sum of £14 17s. 6d. 17 horses are kept for 7 days. For how many days at the same rate should 27 horses be kept for £74 5s.?

16. 3 reaping machines cut 17 ac. 10 sq. po. of wheat in $3\frac{1}{2}$ hr. At the same rate how long would 7 machines take to cut 22 ac. 3 ro.?

17. A party of 7 persons spend in 7 da. £21 10s. $9\frac{1}{2}$d. How long at this rate of expenditure ought £27 13s. $10\frac{1}{2}$d. to serve 3 persons?

18. In 10 days the expenses of a travelling party of 6 amount to £19 15s. At this rate how many persons could have their expenses paid for 13 da. with £42 15s. 10d.?

19. 8 ac. 2 ro. 5 sq. po. of wheat are cut by 3 reaping machines in $1\frac{3}{4}$ hr. At this rate how many machines must be employed to cut 68 ac. 1 ro. in half an hour?

20. For £19 19s. a bale of goods weighing $14\frac{1}{4}$ cwt. is conveyed 21 mi. At the same rate what distance ought a parcel weighing $3\frac{1}{2}$ stones to be carried for £3 9s. 5d.?

21. Four bricklayers build 268.8 cub. ft. of wall in 7 hr. How long at this rate would three of them take to build 316.8 cub. ft.?

22. The gas consumed in $17\frac{1}{2}$ hr. by 126 burners of one size cost £2 5s. $11\frac{1}{4}$d. At this rate how many of the same burners might be in use during 60 hours for £5? And what would be the expense of allowing 9 of them to burn 1 hr. 20 min.?

23. The sum of £100 was invested, and the gain upon it in three

years was £10 10s. What sum invested at this rate for $4\frac{1}{8}$ years would gain £130 3s. $1\frac{1}{2}$d.?

24. A company of men subscribe £12560, and after 17 months of trading with it, the profits are found to amount to £333 12s. 6d. What rate of gain is this per £100 per year?

25. Twelve men build 8100 cub. ft. of masonry in 6 days working 9 hours a day. At the same rate how long would 8 of them take to build 1656 cub. ft.?

26. A vessel 3 ft. broad, 2 ft. deep, and 26 ft. $8\frac{1}{12}$ in. long, holds when full 1000 gall. How many gallons could a vessel hold which is 23 in. long, $17\frac{1}{2}$ in. broad, and .64 in. deep?

27. Also, what must be the length of a vessel, which is to be $7\frac{1}{8}$ in. deep and $11\frac{1}{4}$ in. broad, in order that it may contain 13 gall. 1 qt. 1 pt.?

28. For the use of £375 for 3 years a lender was to receive £45, but instead of taking this when it was due he lent it also at the same rate of charge. What total sum ought he to receive when the loan is repaid 4 years afterwards?

29. In $37\frac{1}{3}$ hr. 17 stone-breakers break stones enough to make a heap $79\frac{1}{3}$ yd. long, 4 ft. broad, and 3 ft. high. At this rate how long would 8 of them take to break as many as would form a heap $24\frac{1}{2}$ yd. long, $2\frac{1}{2}$ yd. broad, and $2\frac{1}{2}$ yd. high?

30. A heap of stones $40\frac{1}{3}$ yd. long, 1 yd. high, $1\frac{2}{9}$ yd. broad, have been broken by 11 men in $80\frac{5}{8}$ hours. At the same rate how many would be required to break stones enough in 112 hours to make a heap $79\frac{1}{3}$ ft. long, $1\frac{1}{3}$ yd. broad, and 1 yd. high?

31. Eleven iron rods, each $6\frac{1}{2}$ ft. long, $1\frac{1}{2}$ in. broad, 2 in. thick, weigh in all 6 cwt. 1 qr. 25 lb. $1\frac{8}{600}$ oz. At this rate what would be the weight of 7 others, each of which is $8\frac{1}{3}$ ft. long, 1.2 in. broad, 1.2 in. thick?

32. The total weight of 3 bars of platinum, each 6.5 in. long, .4 in. broad, .35 in. thick, is 30.80805 oz. avoird. How many bars, each 8 in. long, .55 in. broad, .45 in. thick, could be made from 13.9651875 lb. avoird. of the same metal?

33. 879 deaths occurred in Glasgow in three weeks of February, 1874. Assuming the population to be 508000, calculate the *death-rate* for the period referred to, *i.e.*, the number of deaths per year per thousand of the population.

150. The subjects (workmen's wages, travelling expenses, freight of goods, &c.) with which the foregoing exercises on *rates* deal are expected to be familiar to the learner, so that little chance of error is possible in reasoning about them. With other subjects, however, it is advisable to proceed more cautiously. For example, suppose the following question were proposed.

Example 1. A circular plate of silver 4 in. in diameter weighs $5\frac{1}{3}$ oz. At this rate what would be the weight of another plate of the same thickness, but 7 inches in diameter?

Here probably the learner would at once proceed as follows:—

	Weight of plate	4 in.	in diameter	$=5\frac{1}{3}$ oz.
$\therefore$	,,	1 in.	,,	$=5\frac{1}{3}$ oz. $\div 4$
and $\therefore$	,,	7 in.	,,	$=5\frac{1}{3}$ oz. $\times \frac{7}{4}$
				$=9\frac{1}{3}$ oz.

But this would be incorrect. Instead of the divisor in the second line being 4, it should be 16, that is, 4^2; for we learn from geometry that *a circle* 1 *inch in diameter is not* $\frac{1}{4}$ *but* $\frac{1}{16}$ *of the size* (*area*) *of a circle* 4 *in. in diameter.* Similarly, in the next line the multiplier is not 7 but 49, that is 7^2: the result thus being

$$5\tfrac{1}{3} \text{ oz.} \times \tfrac{49}{16} \text{ or } 16\tfrac{1}{3} \text{ oz.}$$

If the plates, instead of being circular with the lengths of the diameters given, had been *square* with the lengths of the *sides* given, we should have proceeded in exactly the same way; for what has been asserted from geometry regarding circles is but a single case of a widely general law; and as the learner may readily convince himself of its truth in the case of squares he should not fail to do so.

When one magnitude is related to another in this way, viz., so that when we increase the second through multiplication by any number, the first is correspondingly increased through multiplication by the *second power* of that number, then the first magnitude is said to be *directly proportional to the second power of the other*. Thus we say that the area of a square is directly proportional to the second power of the side, or the second power of the diagonal; and the area of a circle directly proportional to the second power of the diameter, the second power of the radius, or the second power of the circumference. This, it should also be remem-

bered, is the relation existing between the *distance* traversed in vacuo by a body from the moment it is let fall towards the earth or any other planet, and the *time* during which it has been falling; that is to say, the distance traversed in the first *two* seconds is *four* times the distance traversed in the first second, and the distance traversed in the first *three* seconds is *nine* times the distance traversed in the first second, and so on.

The numerous pairs of magnitudes we have hitherto had examples of are, as we know, not related to each other in this way, the multiplier being the same for the second magnitude as for the first. If the *quantity* of flour bought be increased three times, the *price* is increased three times; if the number of *machines* be quadrupled, the *work* done is quadrupled; and so of the others. In these cases we say that the former magnitude is *directly proportional to the first power* of the latter, or, simply, is *proportional to* the latter.

Similarly, we may have one magnitude proportional to the *third power* of a second, as in the case of the *bulk of a sphere* or ball and its *diameter*, or the *bulk of a cube* and the *length of its edge*.

Further, two magnitudes may be so related that when we increase the second by multiplication, the first, instead of being increased along with it, is found to be *diminished by division*. Here the magnitudes are said to be *inversely proportional*. Thus the *number of men* employed on a work is inversely proportional to (the first power of) the *time* taken to complete it; the *attraction* between two bodies is inversely proportional to the second power of the *distance* between them (*i.e.*, between their centres of gravity); and so on.

Instances of even more complicated relationships than those here given may be found; so that when two magnitudes unfamiliar to the learner require to be dealt with, his first duty must always be to make inquiry as to the exact relation existing between them.

Example 2. If two bodies 12 miles asunder attract each other with a force of 100 units, what will the attractive force be when they are only five miles separate?

$$
\begin{array}{llll}
 & \text{At the distance of 12 miles the force} & = 100 \text{ units} \\
\therefore & \quad ,, \qquad\qquad 1 \text{ mile} \qquad ,, & = 100 \text{ units} \times 12^2 \\
\text{and } \therefore & \quad ,, \qquad\qquad 5 \text{ miles} \qquad ,, & = 100 \text{ units} \times \frac{12^2}{5^2} \\
 & & = 576 \text{ units.}
\end{array}
$$

Example 3. Five balls of soap $7\frac{6}{7}$ in. in circumference weigh $2\frac{1}{7}$ lb. At this rate, find the weight of a dozen balls of the same soap which measure $4\frac{1}{6}$ in. round.

$$
\begin{aligned}
\text{Required weight} &= 2\tfrac{1}{7} \text{ lb.} \times \frac{12}{5} \times \frac{(4\frac{1}{6})^3}{(7\frac{6}{7})^3} \\
&= \frac{15}{7} \text{ lb.} \times \frac{12}{5} \times \frac{\frac{15625}{216}}{\frac{166375}{343}} \\
&= \frac{3}{7} \text{ lb.} \times \frac{12}{1} \times \frac{15625}{216} \times \frac{343}{166375} \\
&= 3 \text{ lb} \times \frac{1}{1} \times \frac{125}{18} \times \frac{49}{1331} \\
&= \frac{6125}{6986} \text{ lb.}
\end{aligned}
$$

Exercises. Set XCIII.

1. The area of a circle whose diameter is 1000 in. long is found to be 785397.5 sq. in. Find at this rate the area of a circle 200 ft. in diameter.

2. Calculate the value of a square plate of gold of uniform thickness and 8 in. long on each side, from knowing that a square foot of the same plate is worth £13 4s.

3. A body starting from rest takes 10 sec. to fall in vacuo to the earth, and in the first 2 sec. traverses 64.4 ft. How far does it fall altogether?

4. If from a cube of earth whose edge is 4 yd. long the same cart can be filled 108 times, how often could it be filled from a cube 8 ft. on the edge?

5. The rope which tethers a goat is lengthened from 15 ft. to 20 ft. How much more surface has he now to browse over, supposing that formerly he had $706\frac{97}{113}$ sq. ft.?

6. A metallic ball 3.4 in. in diameter is worth £1 6s. 9d. Calculate at this rate the value of a ball of the same metal whose diameter is 1.7 in. longer.

7. A body falling in vacuo towards the earth passes over 4.9 metres in the first second of its fall. How high was it at first if it took $4\frac{1}{2}$ sec. to fall?

8. In the first 3 sec. of its fall in vacuo towards the earth a ball passes over 144.9 ft. What distance will it go in the next 2 sec.?

9. When one magnitude is represented by 1, 2, 3, &c., another is represented by 15, $7\frac{1}{2}$, 5, &c. How are they related to each other?

10. Give instances of a magnitude inversely proportional to (the first power of) another magnitude.

11. Two magnitudes are so related to each other that when the first is represented by 2, 3, 4, &c., the second is represented by $\frac{1}{8}$, $\frac{1}{27}$, $\frac{1}{64}$, &c. What is the relationship between the two?

12. The radius of a circle is .5 in., and the circumference 3.14159 in. Find from this the circumference of a circle of 4.5 in. radius.

13. A book 5.5 in. from a centre of light is held parallel to a wall, the shadow on which is found to be 400 sq. in. in size. How much would this shadow be diminished if the book were brought $2\frac{1}{2}$ in. nearer?

14. Puck says, "I'll put a girdle round about the earth in forty minutes." At this rate how long would he take to do the same for the moon, whose radius is 1080 miles, while the earth's is 7925?

15. The material needed to gild a ball $9\frac{1}{8}$ in. in diameter costs 7s. 10d. What would be the cost at this rate in the case of a ball $3\frac{1}{2}$ in. in diameter?

16. Half-a-dozen hemispherical cups, each 2.5 in. in diameter (internal), hold in all 56 shillings' worth of mercury. What would be the value of the mercury necessary to fill a dozen and half similar cups 2 in. in diameter?

17. 9 cubical blocks of granite, whose edge is $2\frac{1}{2}$ ft. long, are polished on four sides for £30. At this rate what would be the expense of polishing on three sides 8 cubical blocks, whose edge is half a foot longer?

151. PERCENTAGE.—There is a particular way of specifying a rate which is very common in practical life, viz., announcing it as *so much a hundred*. The word *hundred* is not used, but in its stead the equivalent Latin word *centum*, and this commonly in the contracted form *cent.* Thus "5 *per cent.*" means "£5 for every £100," "5 men for every 100 men," "5 tons for every 100 tons," &c., the denomination *pounds*, or *men*, or *tons*, &c., being understood in each particular case from the context. A rate spoken of in this way is called a *percentage*.

(I.) As instances of such, there may be mentioned in the first place certain *wages*, *fees*, &c. For example, a commercial firm, instead of paying a salesman or traveller a regular

wage, may, for obvious reasons, prefer to give him a percentage on the price of the goods he sells; a wage of 1 *per cent. on sales* meaning £1 for every £100 worth of goods sold, or, what is the same thing, 1s. for every 100s. worth, or 1d. for every 100d. worth, &c. *Brokerage*, the fee of a broker, is spoken of in the same way: so also *commission*, the fee of a commission agent; *interest*, the remuneration paid to a money-lender; *premium*, the annual payment to an insurance company; *discount*, a deduction made by tradesmen from a debt, &c.

Example 1. A certain stockbroker's charge is $\frac{3}{4}$ per cent. What will he receive in a money transaction of £1426?

For a transaction involving £100 he receives £$\frac{3}{4}$

$\therefore$ " £1 " $\frac{£\frac{3}{4}}{100}$

and $\therefore$ " £1426 " $\frac{£\frac{3}{4} \times 1426}{100}$

i.e., £10 13s. $10\frac{3}{4}$d. $\frac{1}{5}$

The calculation here necessary is usually performed as follows:—

$$\begin{array}{r}
£1426 \\
\underline{3} \\
4)\underline{4278} \\
10.695 \\
\underline{20} \\
13.90 \\
\underline{12} \\
10.80 \\
\underline{4} \\
3.20
\end{array}$$

From this example it is easily seen that in every case where a percentage upon a sum of money is to be calculated, *we multiply the sum by the rate per cent. and divide by* 100.

Example 2. An agent collects debts to the amount of £620 16s. 8d., from which he is allowed to deduct $2\frac{1}{2}$ per cent. for his trouble. What sum does he thus receive?

$$\text{Sum required} = £2\tfrac{1}{2} \times \frac{£620\ 16s.\ 8d.}{£100}$$
$$= £1552\ 1s.\ 8d. \div 100$$
$$= £15\ 10s.\ 5d.$$

Example 3. The premium of insurance upon a property valued at £8560 is £235 8s. What percentage is charged by the insurance company?

The same question more simply put is, "If the charge for insuring £8560 be £235 8s., what is the charge for insuring £100?" and this the learner should now have no difficulty in answering.

$$\text{Charge for insuring } £8560 = £235\ 8s.$$
$$\therefore \quad \text{"} \quad £100 = £235\ 8s. \times \frac{100}{8560}$$
$$= £1177 \div 428$$
$$= £2\tfrac{3}{4}.$$

Example 4. An agent's commission at $2\frac{1}{2}$ per cent. amounts to £18 1s. 6d. On what sum was the commission allowed?

$$£2\tfrac{1}{2} \text{ is commission on } £100$$
$$\therefore £18\ 1s.\ 6d. \quad \text{"} \quad \text{"} \quad £100 \times \frac{£18\ 1s.\ 6d.}{£2\frac{1}{2}}$$
$$\textit{i.e.}, £1807\ 10s. \div 2\tfrac{1}{2}$$
$$\textit{i.e.}, £803\ 6s.\ 8d.$$

Example 5. A person wishes to insure a house worth £3200 for such a sum as would cover both the value of the house (were it destroyed within the year) and the premium paid at the rate of 4 per cent. to the insurance company. Find this sum.

To obtain £100 he must insure for £100
i.e., to cover premium and £96 " "
∴ for every £96 of the house's value " "

$$\text{and } \therefore \text{ the sum required} = £100 \times \frac{3200}{96}$$
$$= £3333\ 6s.\ 8d.$$

Exercises. Set XCIV.

1. A draper's assistant receives $2\frac{1}{2}$ per cent. of the money drawn for sales. What will his week's wages be if he draws £85 10s.?

2. A house is insured for £5650, the annual payment (premium) being at the rate of ⅜ per cent. What does this payment amount to?

3. B has to pay A a debt of £320 10s., on which, however, A allows discount at 5 per cent. How much less has B to pay?

4. If a commission agent makes a deduction of 1¼ per cent. from £210 6s. 8d. what remains?

5. A gentleman purchases an estate through an agent, the price being £7250, and the agent's commission ⅛ per cent. on the price. What was the total cost to the buyer?

6. Complete the following account of a broker's transactions for a week.

Date.		Business.	Brokerage.			
Nov.	8	Sale of plate, £404 6s. 8d. . .	1¼%			
,,	9	Purchase of furniture, £500 . .	2½%			
,,	10	Auction sale, £66 13s. 4d. . .	4%			
,,	11	,, ,, £20 10s. . . .	5%			
,,	12	Rare china bought, £120 15s. .	2%			
,,	13	Week's shop sales, £60 3s. 4d. .	10%			
		Total				

7. Deduct 5 per cent. from £225 10s. 10d. and 5 per cent. from the remainder.

8. From £225 10s. 10d. subtract 10 per cent. of it.

9. A person insures his life for £6000, two-thirds of it in one office at a premium of £2 10s. 6d. per cent., and the remainder in another at a premium of £2 9s. 9d. per cent. What is his yearly payment?

10. A merchant allows certain debtors a discount of 1 per cent. on the sums they owe him, and the agent who collects the money receives ⅛ per cent of it. If the debts amount to £8000, what sum goes to the merchant?

11. From a debt of £226 5s. the sum of £9 1s. was deducted. At what rate per cent. was discount allowed?

12. In a certain shop the goods are sold at 10 per cent. above cost price, and the salesman receives from his employer 3½ per cent. on the selling price. How much does the employer gain on a quantity of goods sold for £652 10s.?

13. £38 5s. is what percentage of £850?

14. Calculate the expense of insuring a ship valued at £20500, the premium charged being 2½ per cent., the policy duty being ⅛ per cent., and the broker's fee ⅛ per cent.

15. An agent sells a quantity of goods for £660, of which he pays to his employer £658 7s. What percentage has he deducted?

16. The premium charged for insuring a cargo valued at £5625 is

5 guineas per cent. Find the total cost of insuring, allowing for policy duty at 4s. per cent., and agent's commission at 2s. 6d. per cent.

17. After discount at 4 per cent. had been deducted from a debt, the sum remaining to be paid was £197. Find the debt.

18. A manufacturer makes a quantity of cloth at an expense of £1170, and sells it for 15 per cent. more to a clothier, who in turn sells it for £1547 6s. 6d. What percentage on his outlay does the latter thereby receive?

19. An auctioneer, after deducting his commission of $2\frac{1}{2}$ per cent. on the proceeds of a sale, returns to his employer £672 15s. Calculate the sum drawn at the sale.

20. For what sum must a house worth £1161 be insured at $3\frac{1}{4}$ per cent. so that in the case of total loss within the year the owner may recover both the premium and the value of the house?

21. A gentleman pays a life insurance company an annual premium of £41 12s. Knowing that for his age the premium charged is at the rate of £2 13s. 4d. per cent., find the sum for which his life is insured.

22. What premium, at the rate of $2\frac{1}{8}$ per cent., must be paid for the insurance of goods worth £3171 3s., so that in the case of total loss within the year the insurer may recover both the premium and the value of the goods?

23. If from a certain sum 5 per cent. be deducted, and $3\frac{1}{3}$ per cent. from the remainder, there results £3682 5s. 7d. Find the original sum.

24. For what sum must a warehouse worth £96625 be insured at £$1\frac{1}{8}$ per cent., so that, were it totally destroyed in the third year after, the owner might recover the value of the house and the premiums paid?

(II.) Again, this way of stating a rate is very often employed in giving the proportion which the parts of some composite thing form of the whole. For example, if in the examination of a school of 450 scholars 423 passed and 27 failed, we might indicate *the rate of failure* by saying it was "27 in 450," or "9 in every 150," or "3 in every 50"; but, instead of this, the usual form is "so many in every hundred," and there being 6 here in every 100, we say that the rate of failure was "6 per cent.," or that 94 per cent. passed. Similarly we speak of the sick rate in a certain regiment being 12 per cent., of there being 53 per cent. of pure iron in a certain ore, and so on.

Example 1. The milk sold by a dairyman is found on analysis to yield 12 per cent. of water and 3.2 per cent. of

an unknown ingredient. How much of each of these must there be in 12500 gallons of his milky compound?

$$\text{Quantity of water} = 12500 \text{ gall.} \times \frac{12}{100} = 1500 \text{ gall.}$$

$$\text{Quantity of the other} = 12500 \text{ gall.} \times \frac{3 \cdot 2}{100} = 400 \text{ gall.}$$

Example 2. In a town with a population of 22500, 8000 could read and write, and 6140 could read but not write. Find what percentage of the population could read, and what percentage of the readers could write.

$$\text{Total no. of readers} = 8000 + 6140 = 14140.$$

$$\therefore \text{ 1st percentage} = 14140 \times \frac{100}{22500}$$
$$= 62.8\ldots$$

$$\text{and 2nd percentage} = 8000 \times \frac{100}{14140}$$
$$= 56.57\ldots$$

Example 3. Of 240 boys in the higher department of a school 95 per cent. pass an examination in arithmetic, and of 175 in the lower department 92 per cent. pass. What percentage of the whole number of boys passed?

$$\left.\begin{aligned} 95 \text{ per cent. of } 240 &= 240 \times \frac{95}{100} = 228 \\ \text{and } 92 \text{ per cent. of } 175 &= 175 \times \frac{92}{100} = 161 \end{aligned}\right\}$$

$$\therefore \text{ total no. passed out of 415 boys} = 389,$$

$$\text{so that percentage required} = 389 \times \frac{100}{415}$$
$$= 93 \cdot 7 \ldots .$$

The operations here performed are given by the expression

$$\left(\frac{240 \times 95}{100} + \frac{175 \times 92}{100}\right) \times \frac{100}{415};$$

this, however, is the same as

$$\frac{240 \times 95 + 175 \times 92}{415},$$

from which, as involving fewer operations, the result can be more easily obtained.

EXERCISES. SET XCV.

1. How much per cent. is 13 per score? 1 in 10? 1 to the dozen? $1\frac{1}{2}$ out of 25?

2. In a mixed school there are 3 boys to every 2 girls. What percentage of the pupils do the girls form?

3. Our silver coins are made of a metal of which 37 parts in 40 are pure silver and 3 parts copper. What percentage of the metal is silver?

4. "In our first engagement there were two dozen blacks to every Englishman; in our second blacks and whites were as 9 to 1." What percentage of the combatants in each case were blacks?

5. Complete the following table:—

Class.	No. of pupils on roll.	Present on 23rd Dec.	
		Actual No.	Percentage.
First . .	220	202	
Second . .	232	202	
Third . .	156	140	
Fourth . .	125	115	
Fifth . .	65	60	
Sixth . .	24	24	
Total			

6. Brown receives 75 per cent. of the profits of the firm of Brown, Robinson & Co., Robinson 15 per cent., and the other partner the remainder. For the past year the total profits amounted to £6126. Calculate the share of each partner.

7. The deaths from febrile diseases in the city of Glasgow in 1874 were as follows:—typhus fever, 114; relapsing fever, 7; enteric fever, 202; simple continued fever, 10; infantile remitting fever, 11; rheumatic fever, 20. What percentage does each of these form of the total?

8. From a waggon-load of iron ore weighing $6\frac{1}{2}$ tons, there is produced 2 tons 5 cwt. 2 qr. of the pure metal. What percentage of the ore is iron?

9. In a factory where 225 men are employed, 4 per cent. of the workers receive £2 per week, 16 per cent. receive £1 5s., 30 per cent. receive 18s., and the rest 15s. 6d. What sum is necessary to pay them weekly?

10. A hundredweight of gunpowder is made from $85\frac{8}{25}$ lb. of nitre, $15\frac{17}{25}$ lb. of charcoal, and $11\frac{1}{5}$ lb. of sulphur. What percentage does each ingredient form of the whole?

11. A bell is made of 7 cwt. 2 qr. $10\frac{1}{2}$ lb. of copper, and 1 cwt. 3 qr. $3\frac{1}{2}$ lb. of tin. What percentage of the whole is copper, and what percentage tin?

12. Proof spirit contains 49 per cent. of absolute alcohol. In $12\frac{3}{4}$ gallons of proof spirit how much is not alcohol?

13. In a written examination for entrance to the army there are 625 candidates, of whom 375 pass, but of these only 360 are accepted by the medical officer. What percentage failed in the written examination, and of those who passed what percentage did the medical officer reject?

14. In $12\frac{1}{2}$ tons of sea water there is 9 cwt. of saline matter, and of this 2 qr. $5\frac{3}{8}$ lb. is magnesia. What percentage of sea water is saline matter, and what percentage is magnesia?

15. Complete the following table:—

Name of mine.	Output.	Pure copper.		Landowners' allowance of copper (5 per cent.)
		Percentage.	Actual quantity.	
Molul . .	275 tons	15		
Garibaldi .	314 tons	12.25		
A 1 . . .	760 tons	38.6		
Total....				

16. In a school of 472 boys 430 passed an examination in arithmetic, and in another of 312 boys only 212 passed. What percentage of the total number in the two schools passed?

17. In a school of 265 pupils 75 per cent. passed the government examination, and in another of 650 pupils 592 passed. What percentage of the total number of pupils failed to pass?

18. In one factory employing 1160 adults 70 per cent. are women, in another employing 680 adults the percentage is 65. Find what percentage of the whole employés are men?

19. Besides the two factories mentioned in the preceding exercise, there is in the same town another, in which 550 men are employed and no women. Calculate what percentage of the employés of the three factories are women?

20. 2 lb. of sugar and 3 lb. of water are mixed, and it is known that oxygen forms by weight 49.85 per cent. of the sugar and 88.89 of the water. Find what percentage of the mixture is oxygen.

21. In the examination of a class of 35 boys, 3 got 175 marks, 8 got 160, 11 got 110, 13 got 80, the highest number of marks possible being 215. What percentage of boys obtained more than 40 per cent. of the marks?

22. A father leaves all his property to his 3 sons, the first receiving twice as much as the second, and six times as much as the third. What percentage of the property does each receive?

23. 66 per cent. of the land of a farm is cultivated, and of this 12.5 per cent. is planted with potatoes. Knowing that the potatoes occupy 12 ac. 1 ro. 20 sq. po., calculate the size of the farm.

24. A basketful of apples is divided among three boys, A receiving half as much again as B, who receives half as much as C. What percentage of the apples is apportioned to each?

25. In a cotton mill men, women, and children are employed, the total number being 875. The men constitute 16 per cent. of the whole, and 35 per cent. of the adults. How many women are there?

(III.) In the third place, rates of *increase* and *decrease* are put in the form of a percentage. For example, if the number of pupils enrolled in a school in one year be 450 and in the next 477, the *absolute increase* being thus 27, then the *rate of increase* is "27 on 450," or "3 on every 50," or "6 per cent." Again, if a merchant buys a quantity of goods for £500 and sells them for £475, so that the *absolute loss* is £25, the *rate of loss* is "£25 on £500," or "5 per cent." A rate of increase or decrease, as the learner will thus see, tells what proportion the absolute in crease or decrease is of *the original quantity.*

Example 1. Butter is bought at 1s. 3d. per lb. and sold at 1s. 6d. What is the rate per cent. of profit?

$$\text{Gain on } 15\text{d.} = 3\text{d.}$$
$$\therefore \quad ,, \quad 1\text{d.} = \frac{3}{15}\text{d.}$$
$$\text{and } \therefore \quad ,, \quad 100\text{d.} = \frac{3 \times 100}{15}\text{d.} = 20\text{d.}$$

i.e., there is 20 per cent. of profit.

Example 2. By selling cloth at 1s. per yard a draper suffers a loss of 25 per cent. Find the cost per yard to the draper.

$$\text{What is sold for } 75\text{d. cost } 100\text{d.}$$
$$\therefore \quad ,, \quad 1\text{d. cost } \frac{100}{75}\text{d.}$$
$$\text{and } \therefore \quad ,, \quad 12\text{d. cost } \frac{100}{75}\text{d.} \times 12$$

i.e., 16d.

Example 3. Cloth is bought at 1s. 4d. per yard. How must it be sold so that a gain of 50 per cent. may be effected?

What cost 100d. must be sold for 150d.

$\therefore$ " 1d. " $\frac{150}{100}$d.

and $\therefore$ " 16d. " $\frac{150 \times 16}{100}$d.

i.e., 24d.

Example 4. Twenty-eight gallons of wine cost a restaurant keeper 18 guineas. At what price per pint must he sell it so as to gain $7\frac{1}{2}$ per cent.?

$$\text{Selling price of 28 gall.} = £18\ 18\text{s.} \times \frac{107\frac{1}{2}}{100}$$

$$\begin{aligned} \therefore \quad " \quad 1 \text{ pint} &= \frac{£18\ 18\text{s.} \times 107\frac{1}{2}}{100 \times 28 \times 8} \\ &= \frac{378\text{s.} \times 215}{100 \times 28 \times 8 \times 2} \\ &= \frac{189\text{s.} \times 43}{20 \times 28 \times 8} \\ &= \frac{8127}{4480}\text{s.} \\ &= 1\text{s. } 9\tfrac{138}{160}\text{d.} \end{aligned}$$

The selling price per pint would, therefore, probably be fixed at 1s. 10d.

Example 5. By selling cloth at 9d. per yard there occurs a loss of 5 per cent. What should have been the selling price per yard in order to gain 14 per cent.?

Here we may first calculate the prime cost per yard, as in Example 2, and then there remains to be answered a question exactly like that of Example 3:—

$$\text{Prime cost per yd.} = 9\text{d.} \times \frac{100}{95},$$

and knowing that a yard of the cloth cost 9d. $\times \frac{100}{95}$, we ask ourselves for what it must be sold so as to gain 14 per cent., the answer being—

$$\begin{aligned} \text{Selling price per yd.} &= \left(9\text{d.} \times \frac{100}{95}\right) \times \frac{114}{100} \\ &= 9\text{d.} \times \frac{114}{95} \\ &= 9\text{d.} \times \frac{6}{5} \\ &= 10\tfrac{4}{5}\text{d.} \end{aligned}$$

Or, without finding the prime cost, we may reason as follows:—

To bring in £95 for £100 paid, the price per yd. = 9d.

∴ " £1 " " $= \frac{9}{95}$d.

∴ " £114 " " $= \frac{9}{95}$d. × 114

$= \frac{9}{5}$d. × 6

$= 10\frac{4}{5}$d.

Example 6. Biscuits are bought at a certain price per dozen of 13, and sold at the same price per dozen of 12. Find the gain per cent.

Here the relative buying and selling prices not being given, we must try to determine them from the facts which are given. Now,

selling price of 13 biscuits $= \frac{13}{12}$ of the selling price of 12;

∴ $= \frac{13}{12}$ of the buying price of 13;

i.e., $= \frac{1}{12}$ more than the buying price;

and ∴ gain per cent. $= \frac{1}{12}$ of 100

$= 8\frac{1}{3}$.

Example 7. A grocer mixes together 40 lb. of tea which costs him 3s. 9d. per lb., 25 lb. which costs 3s. 2d. per lb., and 20 lb. which costs 3s. per lb., and sells half of the mixture at 3s. 4d. per lb., suffering a loss in so doing. At what price per lb. must he sell the remainder so as to gain 5 per cent. on the whole?

Price of 40 lb. = 3s. 9d. × 40 = £7 10s.

" 25 lb. = 3s. 2d. × 25 = £3 19s. 2d.

" 20 lb. = 3s. × 20 = £3

∴ price of the whole (*i.e.*, 85 lb.) = £14 9s. 2d.

But he wishes to gain 5 per cent. on the transaction;

∴ selling price of the 85 lb. = £14 9s. 2d. × $\frac{105}{100}$

= £15 3s. $7\frac{1}{2}$d.

$$
\begin{aligned}
\text{But selling price of the first } 42\tfrac{1}{2}\text{ lb.} &= 3\text{s. } 4\text{d.} \times 42\tfrac{1}{2} \\
&= £7\ 1\text{s. } 8\text{d.} \\
\therefore \text{ selling price of the second } 42\tfrac{1}{2}\text{ lb.} &= \left.\begin{array}{r} £15\ 3\text{s. } 7\tfrac{1}{2}\text{d.} \\ -\ 7\ 1\text{s. } 8\text{d.} \end{array}\right\} \\
&= £\ 8\ 1\text{s. } 11\tfrac{1}{2}\text{d.} \\
\therefore \text{ required selling price per lb.} &= £\ 8\ 1\text{s. } 11\tfrac{1}{2}\text{d.} \div 42\tfrac{1}{2} \\
&= 3\text{s. } 9\tfrac{38}{85}\text{d.}
\end{aligned}
$$

Exercises. Set XCVI.

1. Apples are sold for half as much as they cost. What is the loss per cent.?

2. Apples are sold for twice as much as they cost. What is the gain per cent.?

3. Apples are sold for half as much again as they cost. What is the gain per cent.?

4. Apples are sold for three-fifths of what they cost. What is the loss per cent.?

5. Goods are bought for £760 and sold for £798. What is the gain per cent.?

6. Cloth which cost 5s. $5\frac{1}{4}$d. a yard was sold for 4s. $8\frac{1}{4}$d. a yard. What was the loss per cent.?

7. A rare book is sold for £8 0s. 6d. which cost the seller £7 15s. What was the gain per cent.?

8. The population of England and Wales in 1861 was 20061725; in 1871, 22704108: of Scotland in 1861, 3061251; in 1871, 3358613: of Ireland in 1861, 5792055; in 1871, 5402759. Find in each case the rate per cent. of increase or decrease.

9. A gentleman buys a house for £2600, spends in improvements on it £265 12s. 6d., and then sells it for £2751. What is his loss per cent.?

10. How must cloth which cost 4s. $8\frac{1}{4}$d. be sold so as to gain 16 per cent.?

11. A piece of cloth containing 405 yd. is bought for £50, and sold at 3s. 4d. per yd. Calculate the gain or loss per cent.

12. A horse which cost £135 was sold at a loss of 10 per cent. What was the selling price?

13. 11 yd. of lace are sold for £11 19s. 3d., the cost price being 18s. 9d. per yd. Find the gain per cent.

14. What would be the population of Great Britain and Ireland in 1881, if the rates of change were to be the same for the decennial period 1871-81 as for 1861-71? (See Ex. 8.)

15. A gold mine is sold at 250 per cent. of profit for £6500. What did it cost the present vendor?

16. Complete the following table of statistics:—

District of Glasgow.	Population in 1861.	Population in 1871.	Increase.		Decrease.	
			Actual.	Percentage.	Actual.	Percentage.
B	45485	60628				
C	36625	38177				
D	29975	26044				
E	28697	33443				
Total..						

17. The population of a town in 1861 was 34800, and the percentage of increase in the next ten years was 101.432. Find its population in 1871.

18. A watch is sold at 22 per cent. of profit, the actual profit being £5 10s. Find the selling price.

19. A quantity of pens is sold by a stationer at 3s. 8d. per gross and he thereby gains 10 per cent. What did they cost him per gross?

20. Envelopes are bought at 6¼d. per thousand, and sold so as to gain 4 per cent. Calculate the selling price per thousand.

21. The percentage of increase in the population of a town in the decennial period 1861-71 was 13.02, and the population in 1871 was 20484. Find the population in 1861.

22. Find the population in 1881 of the town referred to in Exercise 21, on the supposition that the percentage of decrease for the period 1871-81 was the same as that of the increase for 1861-71.

23. Calculate at what price per cwt. 2½ tons of cheese, which cost in all £57 5s. 10d., should be sold so as to gain 12 per cent.

24. 17½ tons of iron are sold at a loss of 5⅝ per cent., the original price per ton having been £4 3s. 4d. What total sum did the vendor receive?

25. 7000 sq. yd. of ground are sold for £875, the vendor making thereby a gain of 25 per cent. What did the ground cost him per acre?

26. A coal agent sells a waggon of coals containing 4 tons 13 cwt. 2 qr. at 9d. per cwt., thereby suffering a loss of 10 per cent. What did the waggon of coals cost him?

27. In 1873 the number of electors in the fifteenth ward of Glasgow was 3608, and in the sixteenth ward 3927; the numbers of the year following showed a decrease of 3.29 per cent. in the fifteenth, and an increase of 4.71 per cent. in the sixteenth. What was the percentage of increase or decrease for the two wards taken together?

28. Tea is sold at 3s. per lb., causing a loss of 4 per cent. At what price per lb. should it have been sold to gain 6 per cent.?

29. By selling velvet at 17s. 3d. per yd. there results a gain of $3\frac{1}{2}$ per cent. What would have been the gain per cent. if the selling price had been 18s. a yd.?

30. By selling cloth at 3s. 2d. per yd. there is occasioned a loss of 5 per cent. At what price should it have been sold so as to gain 5 per cent.?

31. A loss of $2\frac{1}{2}$ per cent. is occasioned by selling cloth at 4s. $0\frac{3}{4}$d. per. yd. Find whether there would have been a gain or a loss if it had been sold at 4s. $1\frac{1}{2}$d., and state the percentage.

32. A quantity of note-paper is sold at $9\frac{3}{4}$d. per lb., the result being a loss of $2\frac{1}{2}$ per cent. What gain or loss per cent. would have resulted from selling it at 10d. per lb.?

33. Knowing that if sugar were sold for $5\frac{1}{4}$d. per lb. the gain would be 5 per cent., find at what price it should be sold so as to gain 10 per cent.

34. A quantity of silk was sold at a loss of 1 per cent., and it is known that had it been sold for 4s. $2\frac{1}{2}$d. per yd. there would have been a gain of 1 per cent. Calculate the actual selling price.

35. An innkeeper buys 50 gall. of whisky at 10s. 6d. per gall., and 20 gall. at 10s. per gall., mixes the two, and adds 5 gall. of water. How must he sell the mixture per gill so as to gain 20 per cent.?

36. In the case of a certain town the percentage of increase of population in 5 years was 25, and in the next 5 years it was 4. What was the percentage of increase for the 10 years referred to?

37. A shopkeeper bought 250 gall. of oil at 2s. 6d. per gall. 5 per cent. of it was lost in transferring to smaller vessels, 48 per cent. of the remainder he sold at 3s. a gallon, and the rest was disposed of at cost price. What did he thus gain or lose per cent.?

38. Marbles are sold at the same price per score as they were bought at per dozen. Find the loss per cent.

39. The income of an institution showed in 1874 an increase of 12 per cent. over the income of 1873, while the income of 1875 showed an increase of only 3 per cent. over that of 1873. Compare in this way the incomes of 1875 and 1874.

40. Silk is bought at a certain price per metre and sold at the same price per yard. Calculate the gain per cent.

41. In 1874 the value of a piece of ground increased 20 per cent., and in 1875 it decreased $18\frac{3}{4}$ per cent. What was the total effect per cent. of these changes?

42. What change per cent. in 1876 would bring the value of the piece of ground referred to in Exercise 41 back to what it was before 1874?

43. Instead of a yard measure, a draper uses a stick which is 36.35 in. long. What does he lose per cent. by so doing?

44. Soap is bought and made up into cakes of the same size, the selling price per cake being the same as the buying price per $\frac{1}{4}$ lb.

What must be the weight of each cake so as to insure a gain of $6\frac{1}{4}$ per cent.?

152. Average Rate.—It is well known that the rate at which anything is done—goods sold, ground ploughed, distance travelled, &c.—may not be the same throughout the whole time occupied in the doing of it; in other words, that the rate may vary. In such a case we cannot, of course, speak of *the* rate at which the thing is done, because there are more than one: still, one single rate is here also often used for convenience, viz., the *average* or *mean* rate, which may be defined as that single imaginary rate which would yield the same final result as the various actual rates. If 20 oxen cost in all £320, *and we know that each of them cost the same sum*, we say that the rate was £16 per ox; but if, on the other hand, some of them were dearer than others, we say that £16 per ox was the *average* rate, or that they cost £16 per head *on the average*. In the latter case it may be that no one of them cost £16, but if this had been the price of each, the total cost would have been as it was, £320. Again, suppose a railway train starts from one station and stops three hours afterwards at another 120 miles farther on, we know that it must have gone at an infinite number of different rates, and, indeed, that from first to last its rate may have been continually varying; we say, however, that the average rate was 40 miles an hour, meaning thereby that had it gone uniformly at this speed the distance would have been accomplished in exactly the same time. In instances like this, where from the nature of the case *average* rate must be meant, it is not uncommon to speak simply of the rate.

Example 1. If I spend £2 10s. on Monday, £2 2s. 5d. on Tuesday, £1 10s. 6d. on Wednesday, 14s. on Thursday, 15s. 6d. on Friday, £3 12s. 9d. on Saturday, and nothing on Sunday, what is my average expenditure per day during the week referred to?

Total expenditure in 7 days =	£2	10	0
+	2	2	5
+	1	10	6
+	0	14	0
+	0	15	6
+	3	12	9
=	£11	5	2

∴ average expenditure per day = £11 5s. 2d. ÷ 7
= £1 12s. 2d.

Example 2. A grocer buys 14 lb. of tea at 2s. 6d. per lb., 15 lb. at 3s. per lb., and 7 lb. at 4s. per lb. Calculate the average price per lb.

Cost of 14 lb. at 2s. 6d. = £1 15
" 15 lb. at 3s. = £2 5
" 7 lb. at 4s. = £1 8

∴ total cost of 36 lb. = £5 8

and ∴ average price per lb. = £5 8 ÷ 36
= 3s.

It should be noted here that were he to mix together what he bought, this would be the value per lb. of the mixture.

Example 3. In 3 weeks a stationer's income from his shop amounts to £47 15s. 6d.; in the next 5 weeks it amounts to £63 12s. 6d.; and in the fortnight following it is £20 14s. Find (1) his average income per day during each of the periods referred to; (2) the average of these averages; and (3) his average income per day over the whole period of 10 weeks.

(1) Average daily income in 1st period = £47 15s. 6d. ÷ 18
= £2 13s. 1d.
" " 2nd " = £63 12s. 6d. ÷ 30
= £2 2s. 5d.
" " 3rd " = £20 14s. ÷ 12
= £1 14s. 6d.

(2) Mean of these 3 averages
= (£2 13s. 1d. + £2 2s. 5d. + £1 14s. 6d.) ÷ 3
= £6 10s. ÷ 3
= £2 3s. 4d.

(3) Average daily income for the 10 weeks

$=(£47\ 15s.\ 6d. + £63\ 12s.\ 6d. + £20\ 14s.) \div 60$
$= £132\ 2s. \div 60$
$= £2\ 4s.\ 0\frac{2}{5}d.$

Here the learner may have expected the last two of the results to be the same; he should, therefore, note that *the average over the whole is not necessarily the same as the mean of the averages over the various parts into which the whole may be divided.*

Example 4. The cost of 24 lb. of sugar was 8s., some of it being bought at $4\frac{1}{2}$d. per lb., and the rest at $3\frac{1}{2}$d. per lb. How much was bought at each price?

The average price per lb. = 8s. ÷ 24 = 4d.

Now the first of the two prices is $\frac{1}{2}$d. more than 4d., and the second is $\frac{1}{2}$d. less; consequently, in order that purchases at the one price may counterbalance purchases at the other, and maintain an average rate of 4d. a lb., equal quantities of the two kinds must be bought. The answer to the question therefore is

12 lb. at the one price and 12 lb. at the other.

Example 5. A coal dealer bought 70 tons of coal, part at 28s. a ton and part at 35s. a ton, the average price per ton being 32s. How much did he buy at each price?

28s. = 4s. less than 32s.
and 35s. = 3s. more than 32s.

∴ cost of 3 tons at 28s. = 12s. less than cost of 3 tons at 32s., and cost of 4 tons at 35s. = 12s. more than cost of 4 tons at 32s.

$\therefore \left.\begin{array}{l}\text{cost of 3 tons at 28s.}\\ \quad\text{and 4 tons at 35s.}\end{array}\right\} = \text{cost of 7 tons at 32s.}$

Hence of every 7 tons bought 3 are at the lower price and 4 at the higher. But the total quantity bought was 70 tons; ∴ there were 30 tons at 28s. and 40 tons at 35s.

Example 6. How must a grocer mix tea at 3s. 3d. a lb. and tea at 2s. $8\frac{1}{2}$d. a lb. so as to form a mixture worth 3s. a lb.?

In putting in the two kinds so as to have a mixture worth 3s. a lb.,

the putting in of 1 lb. at 3s. 3d. causes a loss of 6 halfpence,
and „ „ 2s. 8½d. „ gain of 7 „

Hence, in order that the gain may exactly counterbalance the loss, we must put in 7 lb. of the former for every 6 of the latter; for the loss in so doing would be 6 halfpence × 7, and the gain 7 halfpence × 6, that is, 42 halfpence in both cases. The desired mixture is, therefore, got by taking 6 parts by weight at the higher price for every 7 parts at the lower.

Example 7. How must three different kinds of sugar at 4¼d., 5d., and 5½d. per lb. respectively be mixed so as to give a mixture worth 4¾d. per lb.?

It is clear that if a mixture of the first and second kinds be got worth 4¾d. a lb., and likewise a mixture of the first and third worth 4¾d. a lb., then a mixture of any quantities of these two mixtures must contain the three kinds, and must be worth 4¾d. a lb., and therefore must be the mixture desired. Now proceeding as in Example 6, we find that

	1st Sugar.	2nd Sugar.	3rd Sugar.
In the 1st mixture there must be	1 part	2 parts	—
„ 2nd „	3 parts	—	2 parts
∴ the desired mixture may be formed of	4 parts	2 parts	2 parts

Here we have taken 3 parts of the first mixture to 5 parts of the second; but as we may combine the two mixtures in any manner we choose, it is clear that an infinite variety of answers may be given to the question. For example, taking 6 instead of 3 parts of the first mixture, of which 2 must be of the first sugar and 4 of the second, we should have as our result

5 parts 4 parts 2 parts.

Exercises. Set XCVII.

1. Find the mean of the odd numbers up to and including 15.
2. Of 6 boys the first is 5 ft. 4 in. high, the second is 5 ft. 1½ in., the

third is 5 ft. 6 in., the fourth is 4 ft. 10 in., and the remaining two are the same height, viz., 5 ft. 5 in. What is the height of the boys on an average?

3. A score of sheep cost £2 14s. each, and half a score cost £3 5s. each. What was the average cost per sheep?

4. A butcher buys 3 oxen at £20 each, 5 at £25 10s. each, and 7 at £17 15s. each. What was the average price per ox?

5. Complete the following table of statistics:—

Arrivals of Steam Shipping at Glasgow.			
Year.	No. of Vessels.	Total Tonnage.	Average Tonnage.
1831	7537	545751	
1841	9421	828111	
1851	11062	1021821	
1861	11281	1029480	
1871	12713	1588699	
Total.			

6. A person buys 3 doz. pen-holders at $1\frac{1}{2}$d. each, 4 doz. at $2\frac{1}{4}$d. each, and 1 gross at $\frac{3}{4}$d. each. Find the average price of 1 pen-holder.

7. Find the mean of the second powers of the reciprocals of the even numbers up to and including 8.

8. In a class of 15 boys the average age is $13\frac{1}{12}$ years, in another of 36 boys it is $12\frac{5}{12}$, and in a third of 40 boys it is $10\frac{3}{4}$. Find the gross average (*i.e.*, the average over the whole).

9. A horseman travels for $3\frac{1}{2}$ hours at a speed of 7 miles an hour, then for $2\frac{1}{4}$ hours at 6 miles an hour, and ends by keeping up a speed of 8 miles an hour for 15 minutes. Find his average speed per hour.

10. The average breadth of 5 coins is 1.35 in., of 6 other coins 1.45 in., and of 10 others 1.2 in. Find the difference between the mean of these averages and the gross average.

11. Complete the following meteorological table:—

Date.	Temperature.		Mean Height of Barometer.	Rainfall.
	Maximum.	Minimum.		
Aug. 23	56.9	44.2	29.864	0.03 in.
„ 24	55.1	38.8	30.065	0.00 „
„ 25	59	40	30.012	0.01 „
„ 26	61.9	37.1	30.245	0.00 „
„ 27	62	45.4	30.312	0.00 „
„ 28	57.5	49	30.249	0.00 „
„ 29	63	47	30.197	0.00 „
Average.				

12. In the examination of a class of 35 boys, 3 got 175 marks, 8 got

160, 11 got 110, 13 got 180, the highest number of marks possible being 215. What average percentage of marks was obtained?

13. Thirty boys are enrolled for an arithmetic class, and during 1 week 17 of them have attended 5 days, 5 of them $4\frac{1}{2}$ days, 2 of them 4 days, 1 of them 2 days, and the others have never been present. Find the average number of days attended (1) by each boy who has been present at all during the week, (2) by each boy on the roll.

14. A goldsmith melts together 11 grams of gold and 13 grams of silver. Accounting 5 grams of silver to be worth 1 franc, and gold to be $15\frac{1}{2}$ times more valuable, calculate the value of a gram of the alloy.

15. Pure gold is said to be 24 carats fine: coinage or standard gold which contains 22 parts of pure gold to 2 of alloy is therefore 22 carats fine. Now of the former a goldsmith melts 11 oz., of the latter 3 oz., and 10 oz. of rings and old watch-cases the gold of which is known to be 18 carats fine. How many carats fine is the mixture?

16. 10 oz. of gold alloy 15 carats fine, 7 oz. 10 carats fine, and 12 oz. of unknown fineness are mixed together, the mixture being 12 carats fine. Ascertain the fineness of the 12 oz.

17. How must a grocer mix coffee at 1s. 6d. a lb. and chickory at $5\frac{1}{4}$d. a lb. so that the mixture may be worth 1s. 1d. a lb.?

18. A grain-dealer buys 30 bushels of wheat, some of it at 5s. 6d. a bushel and the rest at 6s. a bushel, the average price being 5s. 10d. a bushel. How much did he buy at each of the prices?

19. How must flour at 2d. a lb. and flour at $2\frac{1}{2}$d. a lb. be mixed so that a stone of the mixture may be worth 2s. 8d.?

20. How many lb. of lard at $7\frac{1}{2}$d. a lb. must be mixed with 63 lb. of butter at 1s. 8d. a lb. so as to make a compound worth 1s. 6d. a lb.?

21. It is desired to mix together two sorts of butter worth 1s. $8\frac{1}{2}$d. and 1s. 11d. a lb. respectively, in order to have a mixture worth 1s. 10d. a lb. Calculate (1) what percentage of the whole must be formed of each sort, (2) how much of the cheaper sort must be taken along with a stone of the dearer, and (3) how much of each there will be in a cwt. of the mixture?

22. In an examination a certain number of boys gain 75 per cent. of the marks obtainable, and the rest gain 58 per cent. Knowing that the number of boys examined was 34, and the gross average percentage gained 65, find how many boys gained 75 per cent.

23. Coffee at 1s. 4d. a lb. is mixed with another sort at 1s. 11d. a lb., and the mixture being sold at 2s. a lb. there results a gain of 20 per cent. What percentage of the mixture did the cheaper coffee form?

24. How must teas at 2s. 6d., 2s. 8d., and 3s. a lb. be mixed so as to form a mixture worth 2s. 9d. a lb.?

25. Coffee at 1s. 5d. a lb., coffee at 1s. 8d. a lb., and chickory at $5\frac{1}{4}$d. a lb. are mixed, the mixture being worth 1s. 6d. a lb. In 35 parts of the mixture how many are there of each of the three ingredients?

26. Type-metal is composed of 16 parts of lead, 4 parts of antimony, and 1 of copper, and is worth 82 centimes per kilogram. Assuming

lead to be worth .55 fr. per kilog., antimony 1.5 fr., and copper 2.42 fr., find another alloy of these metals of the same value as type-metal.

153. VARIOUS WAYS OF EXPRESSING A RATE.—As has already been seen, a rate may be specified in various ways, owing to the variety of units of measurement at our disposal. The mode in use in any particular case may have been fixed by custom, but it should also be noted that in practice this variety is a matter of some convenience. So much per *cent.* may be suitable in the case of large sums of money, but in departments of business where small sums are chiefly dealt with a *poundage*, *i.e.*, a rate of so much per pound, is commonly preferred. Further, the dislike of fractions in ordinary life may have influence in the choice of the mode of expression. When the number of deaths in a year out of every hundred inhabitants is stated for several towns, the result not uncommonly is 2.5, 2.2, 2.6, &c., numbers which cannot be compared unless attention is paid to the fractional parts. If, however, the death-rate is expressed as so many per *thousand*, this difficulty (if it be one) is avoided; and such is actually the mode in use. The exercises which follow are intended to give the learner practice in passing readily from one mode of expressing a rate to another.

Example 1. Express the speed of 44 ft. per second as so many *miles* per *hour*.

$$
\begin{aligned}
44 \text{ ft. per second} &= (44 \times 60 \times 60) \text{ ft. per hour} \\
&= \frac{44 \times 60 \times 60}{3 \times 1760} \text{ mi. per hour} \\
&= \frac{1 \times 60 \times 6}{3 \times 4} \quad \text{,,} \\
&= 30 \quad \text{,,}
\end{aligned}
$$

Example 2. Express a rate of $12\frac{1}{2}$ per cent. per annum as so many *pence* per *pound* per *month*.

$$
\begin{aligned}
£12\tfrac{1}{2} \text{ per cent. per annum} &= \frac{£12\frac{1}{2}}{100} \text{ per pound per annum} \\
&= \frac{£12\frac{1}{2}}{100 \times 12} \text{ per pound per month}
\end{aligned}
$$

$$\text{which} = \frac{12\frac{1}{2} \times 240}{100 \times 12}\text{d. per pound per month}$$
$$= \frac{1 \times 20}{8 \times 1}\text{d.} \quad \text{,,} \quad \text{,,}$$
$$= 2\tfrac{1}{2}\text{d.} \quad \text{,,} \quad \text{,,}$$

EXERCISES. SET XCVIII.

1. The education rate of a town is 1⅓d. per pound. What percentage is this?

2. How much per pound is 15 per cent.?

3. What percentage is a poundage of 7½d.? of 2½d.? of 3¾d.?

4. What poundage is the same as ⅝ per cent.? 1⅔ per cent.? 1¼ per cent.?

5. What percentage is booksellers' discount of "2d. off the shilling"?

6. Discount of 25 per cent. is how much "off the shilling"?

7. How much per cent. per ann. is 1½d. per pound per month?

8. How much per pound per month is 10 per cent. per annum?

9. What speed in yards per minute is the same as 591 ft. per second?

10. 40 miles per hour is a speed of how many yards per second?

11. 220 yd. per minute is a speed of how many miles per hour?

12. 8100 cub. ft. per day is how many cubic inches per second?

13. Half-a-pint a day is how many gallons a year?

14. Express the velocity of 40 ft. per second in terms of the *metre* and *minute.*

15. Express the velocity of 13.6 kilometres per hour in terms of the *yard* and *second.*

16. Half-a-crown per ton per mile is how many centimes per quintal per kilometre?

17. "The temperature rose through 50 Fahrenheit degrees in an hour." Express this rate of rise in terms of the *centigrade degree* and *second.*

18. Express a charge of 2.25 francs per millier per kilometre in terms of English units.

19. Express in terms of the *Fahrenheit degree* and *minute* a rise of .754 centigrade degree per second.

20. [A *foot-pound* is the amount of work necessary to raise a pound of matter 1 ft. high; *kilogram-metre* is similarly understood.] Express a rate of 33000 foot-pounds per minute in terms of the *kilogram-metre* and *hour.*

21. [A *pound-degree* is the amount of heat necessary to raise a pound of water through 1 degree Fahrenheit; *kilogram-degree* is similarly understood, the degree, however, being centigrade.] Express "772 foot-pounds per pound-degree" in terms of the *kilogram-metre* and *kilogram-degree.*

154. INTEREST.—Examples 7, 8, 9, 10 of Set XCII. are instances of the calculation of *Interest*, which is the name given to any sum of money paid in return for a loan of money. Such calculations are of very common occurrence, and therefore a considerable number of additional examples will now be given, stated in the shorter language employed in business.

The rate at which interest is paid is specified by mentioning the interest on £100 lent for one year, and the usual mode of expressing it is *so much per cent. per annum;* or *so much per cent.*, "per annum" being understood; or, in writing, *so much* °/ₒ. The sum of money lent, as distinguished from the sum of money received for lending it, is called the *Principal.* The principal, together with the interest due on it after a given time, is spoken of by the lender as what his principal has *become* in the time referred to, or what it *amounts* to at the end of that time.

Two ways of granting interest are in use by money borrowers, viz., (1) interest may be granted only on the principal handed over by the lender, (2) interest may be granted on interest as well; that is to say, at stated times the interest due may be added to the principal, and thenceforward bear interest itself. The terms "simple" and "compound," applicable to these two *modes of reckoning*, are used instead in connection with the word *interest;* so that, in the former case, the principal is said to be lent at *simple interest*, in the latter at *compound interest.* Usually simple interest is meant unless the contrary be stated.

Example 1. Calculate the simple interest on £724 17s. 8d. for $3\frac{1}{2}$ years at 7 per cent. per annum.

Interest on £100 for 1 year = £7

$\therefore$ " £724 17s. 8d. " $= £7 \times \dfrac{£724\ 17s.\ 8d.}{£100}$

and $\therefore$ " " for $3\frac{1}{2}$ yr. $= £7 \times \dfrac{£724\ 17s.\ 8d.}{£100} \times 3\frac{1}{2}$

$= £177\ 11s.\ 11.14d.$

Operations performed :—

£	s.	d.
724	17	8
		7
5074	3	8
		$3\frac{1}{2}$
2537	1	10
15222	11	0
177.59	12	10
20		
11.92		
12		
11.14		

Or, since we know (p. 201) that

Int. on £724 17s. 8d. for 1 yr. $=$ £724 17s. 8d. $\times \frac{7}{100}$

$\therefore$ „ „ $3\frac{1}{2}$ yr. $=$ £724 17s. 8d. $\times \frac{7}{100} \times 3\frac{1}{2}$

$=$ £177 11s. 11.14d.

Example 2. Find the interest on £52 17s. 6d. for 5 years 11 months at $2\frac{1}{8}$ per cent. per annum.

Interest required $=$ £52 17s. 6d. $\times \frac{2\frac{1}{8}}{100} \times 5\frac{11}{12}$

$=$ £6 12s. $11\frac{161}{320}$d.

Operations performed :—

	£	s.	d.
	52	17	6
			$2\frac{1}{8}$
	6	12	$2\frac{1}{4}$
	105	15	0
	112	7	$2\frac{1}{4}$
			$5\frac{11}{12}$
12)	1235	19	$0\frac{3}{4}$
	102	19	$11\frac{1}{16}$
	561	15	$11\frac{1}{4}$
	6.64	15	$10\frac{6}{16}$
	20		
	12.95		
	12		
	$11.50\frac{6}{16}$		

$$\frac{50\frac{6}{16}}{100} = \frac{805}{1600} = \frac{161}{320}.$$

Observe that in the calculation of interest, a month is looked upon as $\frac{1}{12}$ of a year, and as containing 30 days.

Example 3. What will £214 16s. 4d. amount to after lying at interest for 15 weeks, the rate being $2\frac{1}{2}$ per cent. per annum?

$$\text{Interest on £214 16s. 4d. for 1 yr.} = \text{£214 16s. 4d.} \times \frac{2\frac{1}{2}}{100}$$

$$\therefore \quad ,, \quad ,, \quad \text{15 wk.} = \text{£214 16s. 4d.} \times \frac{2\frac{1}{2}}{100} \times \frac{15}{52}$$

$$= \text{£214 16s. 4d.} \times \frac{3}{8 \times 52}$$

$$= \text{£644 9s.} \div 416$$

$$= \text{£1 10s. } 11\tfrac{88}{104}\text{d.}$$

$$\therefore \text{amount required} = \left.\begin{array}{l} \text{£} \quad \text{1 10s. } 11\frac{88}{104}\text{d.} \\ +\text{214 16s. 4d.} \end{array}\right\}$$

$$= \text{£216 7s. 4d. nearly.}$$

Example 4. Find the interest on £12 15s. 6d. for 212 days at 3 per cent. per annum.

$$\text{Interest on £12 15s. 6d. for 1 yr.} = \text{£12 15s. 6d.} \times \frac{3}{100}$$

$$\therefore \quad ,, \quad ,, \quad \text{212 da.} = \text{£12 15s. 6d.} \times \frac{3}{100} \times \frac{212}{365}$$

$$= \text{£12 15s. 6d.} \times \frac{6 \times 212}{73000}$$

$$= \text{£16249 16s.} \div 73000$$

$$= \text{4s. } 5\tfrac{3869}{9125}\text{d.}$$

Here, instead of dividing 212 and 100 by 4, in order to simplify the operations indicated in the second line, we multiply a numerator and denominator by 2, thus obtaining the easy divisor 73000.

Example 5. £8756 12s. is deposited in a bank on 3rd April, 1875, and withdrawn on 2nd January, 1876. Calculate the interest due, the rate being $2\frac{1}{4}$ per cent. per annum.

$$\text{No. of days} = 27 + 31 + 30 + 31 + 31 + 30 + 31 + 30 + 31 + 2 = 274.$$

$$\therefore \text{interest required} = \text{£8756 12s.} \times \frac{2\frac{1}{4}}{100} \times \frac{274}{365}$$

$$= \text{£8756 12s.} \times \frac{4\frac{1}{2} \times 274}{73000}$$

$$= \text{£147 18s. } 0\tfrac{5634}{9125}\text{d.}$$

Here observe that in calculating the number of days for which interest is allowed we count 2nd January and 3rd April as making only one day: or, what is the same thing, we count 2nd January one day, and do not count 3rd April at all, there being thus from April, 30−3 days, and from January, 2 days.

In these calculations it will sometimes be found advantageous to express shillings, pence, and farthings as a decimal fraction of a pound, it being sufficient that the final result be correct to the 3rd right-hand place: and, of course, here, as elsewhere, the learner should be ready to recognise when this, or any other change, is likely to prove useful. For example, the operations of Example 2 may be performed as follows:—

$$
\begin{array}{r}
\pounds 52.875 \\
2\frac{1}{8} \\
\hline
6.609 \\
105.75 \\
\hline
112.359 \\
6-\frac{1}{12} \\
\hline
674.154 \\
9.363 \\
\hline
6.648 \\
20 \\
\hline
12.960 \\
12 \\
\hline
11.52
\end{array}
$$

Exercises. Set XCIX.

Find the simple interest on—

1. £245 for 1 year at 3 per cent. per annum.
2. £376 for 1 year at $2\frac{1}{2}$ per cent. per ann.
3. £148 12s. 6d. for 1 year at 4 per cent. per ann.
4. £260 15s. for 3 years at 2 per cent. per ann.
5. £750 17s. 6d. for 2 years at $2\frac{1}{2}$ per cent. per ann.
6. £190 5s. for $7\frac{1}{2}$ years at 2 per cent. per ann.
7. £2520 6s. 3d. for 8 years at $2\frac{1}{8}$ per cent. per ann.
8. £185 6s. 3d. for $6\frac{1}{4}$ years at 3 per cent. per ann.
9. £370 6s. 3d. for 6 years at $3\frac{1}{4}$ per cent. per ann.

10. £500 7s. 6d. for 5 years at $1\frac{1}{4}$ per cent. per ann.
11. £1241 5s. for 2 years at $1\frac{7}{8}$ per cent. per ann.
12. £611 10s. 10d. for 2 years 6 months at 4 per cent. per ann.
13. £310 0s. 5d. for 3 years 9 months at 2 per cent. per ann.
14. £16 10s. for 1 year 3 months at 4 per cent. per ann.
15. £120 7s. 6d. for 1 year $10\frac{1}{2}$ months at 5 per cent. per ann.
16. £260 4s. 4d. for 17 weeks at $2\frac{1}{2}$ per cent. per ann.
17. £390 6s. 6d. for 15 weeks at $2\frac{1}{8}$ per cent. per ann.
18. £175 10s. for 73 days at $1\frac{1}{4}$ per cent. per ann.
19. £1642 10s. for 109 days at $3\frac{1}{2}$ per cent. per ann.
20. £1916 5s. for 272 days at $3\frac{1}{8}$ per cent. per ann.
21. £1260 17s. 6d. for $3\frac{1}{4}$ years at $2\frac{7}{8}$ per cent. per ann.
22. £217 12s. 6d. for 5 years 5 months at $4\frac{1}{4}$ per cent. per ann.
23. £976 4s. 8d. for 11 weeks at $2\frac{1}{4}$ per cent. per ann.
24. £128 4s. 7d. for 111 days at $3\frac{3}{4}$ per cent. per ann.
25. £400 3s. 9d. for $\frac{1}{4}$ year at $1\frac{7}{8}$ per cent. per ann.
26. £731 10s. 5d. for 2 years 113 days at $2\frac{1}{2}$ per cent. per ann.
27. 600 guineas for 3 years 217 days at 4 per cent. per ann.
28. £54 16s. 4d. for 2 years 7 weeks at £1 7s. 6d. per cent. per ann.
29. 645 francs 25 centimes lent from 13th May to 13th October at $1\frac{1}{4}$ per cent. per ann.
30. £465 17s. 1d. for 1 year 7 months 18 days at $2\frac{1}{2}$ per cent. per ann.
31. The sum of £5600 lies in a bank from 3rd January, 1874, to 20th May, 1875. What interest is due on it at the latter date, the rate being $3\frac{1}{2}$ per cent. per ann.?
32. What would £350 amount to at Christmas, 1875, if lent to a banker on 29th September of the same year, interest being allowed at the rate of $2\frac{3}{8}$ per cent. per ann.?
33. The rate being 4 per cent. per ann., what interest must be due on 12th October, 1875, on a deposit of £360 12s. 6d. which has been in the bank since 5th March, 1874?
34. The sum of £850 17s. was deposited in a bank on 31st May, 1874. What would it amount to on 1st January, 1875, interest being given at the rate of $2\frac{1}{8}$ per cent. per ann.?
35. The sum of £400 15s. 2d. was lent at a charge of $5\frac{1}{2}$ per cent. per ann. from 30th June, 1871, to 1st February, 1873. What total sum should be returned at the latter date?

The foregoing exercises bearing on the subject of interest are of the kind most commonly required in business, viz., where (1) the *principal*, (2) the *time* of loan, and (3) the *rate* are known, and it is desired to find (4) the *interest* due, or the *amount* which the principal has increased to. Any three, however, of these being given, it is possible to find the fourth; and as it is sometimes useful to be able to

do this, we shall now briefly consider the three less important cases that remain, viz., (I.) where the *rate per cent. per annum* is to be found; (II.) where the *time of loan* is to be found; and (III.) where the *principal* or *original sum lent* is to be found.

(I.) Example 1. At what rate per cent. per annum will the interest on £760 for 5 years be £95?

The question in other words is: If £760 gain £95 in 5 years, what would £100 gain in 1 year? And we proceed directly from the known to the unknown, exactly as in Set XCII., Ex. 24, thus:—

$$\begin{array}{llll} & \text{Interest on } \pounds 760 \text{ for } 5 \text{ yr.} & = \pounds 95 \\ \therefore & \quad " \quad \pounds 100 \quad " & = \pounds 95 \times \frac{100}{760} \\ \therefore & \quad " \quad \pounds 100 \text{ for } 1 \text{ yr.} & = \pounds 95 \times \frac{100}{760} \times \frac{1}{5} \\ & & = \pounds 95 \times \frac{1}{38} \\ & & = \pounds 2\frac{1}{2}. \end{array}$$

Example 2. Lying at interest for 156 days £1095 becomes £1118 8s. Find the rate of interest per cent. per annum.

Here we are not told, as in Example 1, the interest gained, but it is at once seen to be £1118 8s.−£1095, *i.e.*, £23 8s., and we thus proceed as before:—

$$\begin{array}{llll} & \text{Interest on } \pounds 1095 \text{ for } 156 \text{ days} & = \pounds 23 \text{ 8s.} \\ \therefore & \quad " \quad \pounds 100 \quad " & = \pounds 23 \text{ 8s.} \times \frac{100}{1095} \\ \therefore & \quad " \quad \pounds 100 \text{ for } 1 \text{ year} & = \pounds 23 \text{ 8s.} \times \frac{100}{1095} \times \frac{365}{156} \\ & & = \pounds 23 \text{ 8s.} \times \frac{25}{3} \times \frac{1}{39} \\ & & = \pounds 585 \div 117 \\ & & = \pounds 5. \end{array}$$

(II.) Example 1. How long must £892 10s. lie in a bank to gain £130 3s. 1½d., interest being given at the rate of 3½ per cent. per annum?

The question in other words is: If £100 gain £3½ in a

year, in what time will £892 10s. gain £130 3s. 1½d.? And starting from what is known, we reason towards the unknown as follows:—

$$\begin{aligned}
&\text{£}100 \text{ gains } \text{£}3\tfrac{1}{2} &&\text{in 1 yr.}\\
\therefore\ &\text{£}892\ 10\text{s. gains } \text{£}3\tfrac{1}{2} &&\text{in 1 yr.} \times \frac{100}{892\frac{1}{2}}\\
\therefore\ &\text{£}892\ 10\text{s. gains } \text{£}130\ 3\text{s. } 1\tfrac{1}{2}\text{d.} &&\text{in 1 yr.} \times \frac{100}{892\frac{1}{2}} \times \frac{\text{£}130\ 3\text{s. } 1\frac{1}{2}\text{d.}}{\text{£}3\frac{1}{2}}\\
& &&\textit{i.e.}, \text{ 1 yr.} \times \frac{200}{1785} \times \frac{62475 \text{ halfp.}}{1680 \text{ halfp.}}\\
& &&\textit{i.e.}, \text{ 1 yr.} \times \frac{5}{357} \times \frac{12495}{42}\\
& &&\textit{i.e.}, \text{ 1 yr.} \times \frac{5}{1} \times \frac{35}{42}\\
& &&\textit{i.e.}, 4\tfrac{1}{6} \text{ yr.}
\end{aligned}$$

Example 2. The rate of interest granted by a bank being 2¼ per cent. per annum, how long must £330 be deposited so as to become £332 9s. 6d.?

To become £332 9s. 6d., a deposit of £330 must gain £2 9s. 6d. of interest. Noting this, we proceed as before:—

$$\begin{aligned}
&\text{£}100 \text{ gains } \text{£}2\tfrac{1}{4} &&\text{in 1 yr.}\\
\therefore\ &\text{£}330 \text{ will gain } \text{£}2\tfrac{1}{4} &&\text{in 1 yr.} \times \frac{100}{330}\\
\text{and } \therefore\ &\text{£}330 \text{ will gain } \text{£}2\ 9\text{s. } 6\text{d.} &&\text{in 1 yr.} \times \frac{100}{330} \times \frac{\text{£}2\ 9\text{s. } 6\text{d.}}{\text{£}2\frac{1}{4}}\\
& &&\textit{i.e.}, \text{ 1 yr.} \times \frac{10}{33} \times \frac{99}{90}\\
& &&\textit{i.e.}, \frac{1}{3} \text{ yr.}
\end{aligned}$$

Example 3. In what time will a sum of money double itself lying at interest at the rate of 4 per cent. per annum?

The question takes for granted that the time will be the same, whatever may be the sum of money deposited; and such is easily seen to be the case. Let us suppose, therefore, that it is £100; then

$$\begin{aligned}
\text{since } &\text{£}100 \text{ gains } \text{£}4 &&\text{in 1 yr.}\\
&\text{£}100 \text{ will gain } \text{£}100 &&\text{in 1 yr.} \times \frac{100}{4}\\
& &&\textit{i.e.}, 25 \text{ yr.}
\end{aligned}$$

(III.) Example 1. Calculate what sum must be at interest

for $2\frac{1}{2}$ years, at the rate of $1\frac{1}{4}$ per cent. per annum, to gain £15 8s. 9d.

£$1\frac{1}{4}$ would be gained in 1 yr. by £100

$\therefore$ £15 18s. 9d. " " $£100 \times \frac{£15\ 18s.\ 9d.}{£1\frac{1}{4}}$

$\therefore$ £15 18s. 9d. " in $2\frac{1}{2}$ yr. by $£100 \times \frac{£15\ 18s.\ 9d.}{£1\frac{1}{4}} \times \frac{1}{2\frac{1}{2}}$

i.e., $£100 \times \frac{£15\ 18s.\ 9d.}{£\frac{25}{8}}$

i.e., $£100 \times \frac{127\frac{1}{2}}{25}$

i.e., £510.

Example 2. What sum of money would become £362 11s. 4d. if laid out at interest for 3 years at the rate of $2\frac{1}{7}$ per cent. per annum?

£100 would gain in 1 yr. £$2\frac{1}{7}$

$\therefore$ £100 " 3 yr. $£2\frac{1}{7} \times 3$

i.e., £$6\frac{3}{7}$

$\therefore$ £100 would become in 3 yr. £$106\frac{3}{7}$

Since then,

sum which becomes £$106\frac{3}{7}$ in 3 yr. = £100

$\therefore$ " " £362 11s. 4d. $= £100 \times \frac{£362\ 11s.\ 4d.}{£106\frac{3}{7}}$

$= £100 \times \frac{£2537\ 19s.\ 4d.}{£745}$

$= £2537\ 19s.\ 4d. \times \frac{20}{149}$

$= £50759\ 6s.\ 8d. \div 149$

$= £340\ 13s.\ 4d.$

This form of the third case is practically more important than any of the six exercises preceding. On this account, and also because the mode of procedure is different, the attention of the learner is more particularly directed to it. It will be found a help to remember that the first aim must be to find *what £100 would amount to in the given time at the given rate*, and that in doing this care must be taken not to proceed as follows:—

£100 would gain in 1 yr. £$2\frac{1}{7}$

$\therefore$ £100 would become in 1 yr. £$102\frac{1}{7}$

$\therefore$ £100 " 3 yr. $£102\frac{1}{7} \times 3$,

in the last line of which there is a serious error of reasoning, the amount to which a sum has increased not being proportional to the time the sum has been lying at interest.

Example 3. When the rate is at 5 per cent. per annum, a certain sum lying at interest becomes £1509 16s. in 249 days. Find it.

In 249 days £100 would become $£100 + £5 \times \frac{249}{365}$

i.e., $£103\frac{30}{73}$.

Then, since

sum which becomes $£103\frac{30}{73}$ in 249 da. $= £100$

$\therefore$ " " $£1509\frac{4}{5}$ " $= £100 \times \frac{1509\frac{4}{5}}{103\frac{30}{73}}$

$= £100 \times \frac{7549}{5} \times \frac{73}{7549}$

$= £20 \times 73$

$= £1460.$

Exercises. Set C.

1. What length of time must £125 lie at simple interest at the rate of 5 per cent. per ann. to gain £75 ?

2. The sum of £790 gains in 3 years £2 19s. of interest. Find the rate per cent. per ann.

3. When the rate of interest is 2½ per cent. per ann., a certain sum gains £63 15s. in 2½ years. Find the sum.

4. In 3½ years the interest gained by £420 amounts to 21 guineas. Calculate the rate per cent. per ann.

5. When the rate of interest in a bank is 3½ per cent. per ann. a deposit of £227 10s. gains £31 17s. How long must it have been in the bank ?

6. What sum must lie at simple interest for 7½ years at the rate of 2 per cent. per ann. in order to gain £57 1s. 6d. ?

7. A sum of £250 6s. 8d. lying at simple interest gains £18 18s. $7\frac{11}{20}$d. when the rate is 2¾ per cent. per ann. How long must it have been in the bank ?

8. What must be the rate of interest per cent. per ann. when £640 becomes £667 4s. in 2 years ?

9. In what time will £725 become £833 15s. if laid out at interest at the rate of 5 per cent. per ann. ?

10. A sum of money is placed at interest for 5 years at the rate of 3¾ per cent. per ann. and gains in that time £85 13s. 3⅜d. What will it then amount to ?

11. A person whose money is invested at $2\frac{1}{2}$ per cent. per ann. has an income therefrom of 1s. a day. Find the amount of the investment.

12. Find in how many days £657 will become £657 15s. $6\frac{81}{25}$d. if placed at simple interest at the rate of $2\frac{1}{2}$ per cent. per ann.

13. Find in what time a sum of money placed at simple interest at the rate of 5 per cent. per ann. will treble itself.

14. The interest on the twenty-fourth part of half-a-crown for 438 days is the thousandth part of a farthing. Calculate the rate per cent. per ann.

15. A person places £30 8s. 4d. at interest at the rate of $2\frac{1}{2}$ per cent. per ann. In how many days will it realise half-a-crown of profit?

16. An exchequer bill for £550 bears $3\frac{3}{4}$d. of interest per diem. What rate of interest is this per cent. per ann.?

17. A sum is placed at interest at the rate of 5 per cent. per ann. on 11th March, and at a certain date it is found to amount to £4076 1s., of which £61 1s. is interest. Find the date.

18. On 4th January, 1876, a sum is placed in a bank, and on 18th November of the same year it amounts to £4952, of which £207 7s. is interest. Calculate the rate of interest.

19. A certain sum with the interest on it for 4 years at the rate of $2\frac{1}{2}$ per cent. per ann. amounts to £72 12s. Find it.

20. Find what sum invested for $3\frac{3}{4}$ years at 5 per cent. per ann. will at the end of that time amount to £542 10s. $9\frac{3}{8}$d.

21. A certain sum, together with the interest on it for 125 days at 4 per cent. per ann., amounts to £222 7s. 6d. Find the sum.

22. What sum placed in a bank on 18th March will amount to £2254 4s. on 18th October, the rate of interest being 5 per cent. per ann.?

23. Calculate what sum of money must be invested for $1\frac{1}{2}$ year at 3 per cent. per ann. in order to become £778 12s. $10\frac{1}{2}$d.

24. On 13th March a sum of £5110 is placed at simple interest, and on 19th October it amounts to £5196 6s. Find the rate of interest per cent. per ann.

25. A sum is placed at simple interest and in 50 years the interest it has gained is double the principal. Find the rate of interest per cent. per ann.

26. On the 29th February a sum of money is put out at interest at the rate of $6\frac{1}{4}$ per cent. per ann., and on the 17th December of the same year it has increased to £306 3s. 9d. Find the sum invested.

27. How much interest must have been gained by a sum which after lying for $7\frac{1}{2}$ years amounted to £285 17s. $9\frac{3}{4}$d., the rate being $2\frac{1}{2}$ per cent. per ann.?

28. A sum is deposited at simple interest at the rate of $2\frac{1}{2}$ per cent. per ann. In what time will the interest due be five-eighths of the deposit?

29. A certain sum is placed at interest at the rate of 4 per cent. per ann., and after 2 years it amounts to £166 14s. 6d. How much of this is interest?

30. A sum of £200 8s. 4d. lies at interest at the rate of $2\frac{1}{8}$ per cent. per ann., and after a certain time it has thereby increased to £234 9s. 9d. How much greater would the increase be if it were allowed to lie 2 years longer?

31. A certain sum placed at simple interest at the rate of 3 per cent. per ann. would gain in 5 years £84 1s. $10\frac{1}{2}$d. What would it amount to in 9 years?

32. A sum of a certain number of pounds was placed at simple interest and in $2\frac{1}{2}$ years was found to amount to the same number of guineas. Find the rate per cent. per ann.

33. A certain sum placed at interest at the rate of $2\frac{1}{2}$ per cent. per ann. has in $3\frac{1}{2}$ years increased to £2474 1s. 3d. What would it have increased to if the rate of interest had been twice as much?

34. A person invests a sum of money at $4\frac{1}{8}$ per cent. per ann., and at the end of $3\frac{1}{2}$ years finds it has increased to £545. Had the sum invested been thrice as great, how many pounds more of interest would he have received?

35. A person deposits a sum in a bank, and after a year deposits as much more, and half a year afterwards withdraws the whole with interest. Knowing that the rate was 4 per cent. per ann., and that the sum finally drawn was £691 17s. 6d., calculate the sum first deposited.

155. If a person deposit a sum in a bank, he may withdraw it, at the end of a year, say, and receive his interest; then, placing both principal and interest in the bank as a new deposit, he may withdraw them at the end of another year along with the interest due, and so on. The possibility of a person acting in this way causes the bankers to offer another arrangement to depositors, viz., to give the second year, of their own accord, interest on the interest due at the end of the first; and this, of course, is what such a depositor as is referred to aimed at getting. When the interest is calculated in this way, at stated intervals, and added to the principal, the sum is said, as we have seen, to be placed at compound interest. The interval is usually a year, or half a year, but may be of any length agreed upon.

Example 1. A sum of £900 is placed in a bank where interest at the rate of 5 per cent. per annum is calculated

yearly, and added to the principal. Find what it will amount to at the end of 3 years.

Principal for 1st yr. $= £900$

$\therefore$ interest „ $= £900 \times \frac{5}{100}$

$= £45$.

Principal for 2nd year $= £900 + 45 = £945$

$\therefore$ interest „ $= £945 \times \frac{5}{100}$

$= £47$ 5s.

Principal for 3rd year $= £945 + £47$ 5s.

$= £992$ 5s.

$\therefore$ interest „ $= £992$ 5s. $\times \frac{5}{100}$

$= £49$ 12s. 3d.

$\therefore$ amount at end of 3 yr. $= £992$ 5s. $+ £49$ 12s. 3d.

$= £1041$ 17s. 3d.

Example 2. A sum of £750 is placed in a bank where interest at the rate of $2\frac{1}{2}$ per cent. per annum is calculated yearly and added to the principal. How much interest will have accrued in 4 years? If the sum had been placed at simple interest at the same rate, what would have been the difference?

Principal for 1st year $= £750$

$\therefore$ interest „ $= £750 \times \frac{2\frac{1}{2}}{100}$

$= £750 \times \frac{1}{40}$

$= £18.75$

Principal for 2nd yr. $= £750 + £18.75$

$= £768.75$

$\therefore$ interest „ $= £768.75 \times \frac{1}{40}$

$= £\ 19.21875$

Principal for 3rd yr. $= £787.96875$

$\therefore$ interest „ $= £\ 19.6992\ldots$

Principal for 4th yr. $= £807.6679\ldots$

$\therefore$ interest „ $= £\ 20.1916\ldots$

$$\begin{aligned}\therefore \text{Total interest} &= £18.75\\ &\quad+19.2187\ldots\\ &\quad+19.6992\ldots\\ &\quad+20.1916\ldots\\ &= £77.8596\ldots\\ &= £77\ 17\text{s. } 2\text{d. nearly.}\end{aligned}$$

Had the sum been placed at simple interest, the principal would have remained the same throughout; consequently the interest for the 2nd, 3rd, and 4th years would have been the same as for the 1st, viz., £18.75, and ∴ would have been in all £18.75 × 4, *i.e.*, £75, so that

$$\text{difference required} = £2\ 17\text{s. } 2\text{d.}$$

Example 3. A sum of £740 18s. 9d. lies at interest for 2 years 115 days at the rate of $3\frac{1}{2}$ per cent. per annum, the interest being added yearly. What does it amount to at the end of that time?

$$\begin{aligned}\text{Interest for 1st yr.} &= £740.9375 \times \tfrac{3\frac{1}{2}}{100}\\ &= \quad 22.228125\\ &\quad+\ \ 3.70468\\ \text{Interest for 2nd yr.} &= £766.8703 \times \tfrac{3\frac{1}{2}}{100}\\ &= \quad 23.006111\\ &\quad+\ \ 3.83435\\ \text{Interest for 115 da.} &= £793.71076 \times \tfrac{7}{200} \times \tfrac{115}{365}\\ &= \quad 3.9685538 \times 7 \times \tfrac{23}{73}\\ &= \quad 8.75256\ldots\\ \therefore \text{amount required} &= £802.4633\ldots\\ &= £802\ 9\text{s. } 3\text{d. nearly.}\end{aligned}$$

In multiplying above by $\frac{3\frac{1}{2}}{100}$ we have mentally shifted the units' mark two places to the left, and multiplied the result by 3, and then by $\frac{1}{2}$.

In the case of Example 1 we might have taken the following instructive mode of procedure.

Since the interest on £100 for 1 year = £5

$\therefore$ " £1 " $= £\frac{5}{100} = £.05$

and, therefore,

in 1 year £1 will amount to £1.05.

Hence,

in 1st year £900 will amount to £900 × 1.05
" 2nd " £900 × 1.05 " £900 × 1.05 × 1.05
" 3rd " £900 × 1.05 × 1.05 " £900 × 1.05 × 1.05 × 1.05
i.e., £900 × $(1.05)^3$
which = £900 × 1.157625
= £1041.8625
= £1041 17s. 3d.

If the rate had been 4 per cent. per annum, and the time 7 years, the result would clearly have been £900 × $(1.04)^7$ instead of £900 × $(1.05)^3$. If the rate had been 2, and the time 6 years, we should have had £900 × $(1.02)^6$, and so on.

Example 4. What will £4000 deposited in a bank amount to in $1\frac{1}{2}$ years if interest at 4 per cent. per annum be added half-yearly to the principal?

Amount required = £4000 × $(1.02)^3$
= £4000 × 1.061208
= £4244.832
= £4244 17s. 7.68d.

Exercises. Set CI.

Find what the following sums will amount to if placed at compound interest, the calculations being made yearly :—

1. £780 ;	rate $2\frac{1}{2}$	per cent.	per ann. ;	time	2	years.
2. £780 ;	" 5	"	"	"	2	"
3. £2220 ;	" 4	"	"	"	2	"
4. £500 ;	" 5	"	"	"	3	"
5. £7000 ;	" 5	"	"	"	5	"
6. £6420 ;	" 3	"	"	"	2	"
7. £540 17s. 6d. ;	" $1\frac{1}{2}$	"	"	"	2	"
8. £9046 12s. 6d. ;	" $2\frac{1}{4}$	"	"	"	3	"
9. £60 14s. 9d. ;	" $2\frac{1}{8}$	"	"	"	3	"
10. £834 18s. 4d. ;	" $2\frac{1}{2}$	"	"	"	4	"

The following sums are placed at compound interest, the calculations

being made half-yearly. Find the total interest gained in each case :—

11. £540 17s. 6d.;	rate $2\frac{1}{2}$	per cent.	per ann.;	time	$1\frac{1}{2}$ year.
12. £623 12s. 6d.;	,, 4	,,	,,	,,	1 ,,
13. £30;	,, $2\frac{1}{2}$	,,	,,	,,	2 ,,
14. £678 12s.;	,, $3\frac{1}{2}$	,,	,,	,,	$1\frac{1}{2}$,,
15. £148 7s. 8d.;	,, 3	,,	,,	,,	3 ,,

Calculate what the following sums will have increased to if placed out at compound interest, the calculations being made yearly in the case of the first two, half-yearly in the case of the second two, and quarterly in the case of the last two :—

16. £420 16s. 3d.;	rate $2\frac{1}{8}$	per cent.	per ann.;	time	2 yr. 7 mo.
17. £908 15s. 4d.;	,, $3\frac{7}{8}$	,,	,,	,,	3 yr. 105 da.
18. £840 10s. 6d.;	,, $2\frac{1}{4}$	,,	,,	,,	1 yr. 8 mo.
19. £660 12s. 2d.;	,, $1\frac{5}{8}$	,,	,,	,,	$9\frac{1}{2}$ mo.
20. £550 17s. 10d.;	,, $4\frac{3}{4}$	,,	,,	,,	$9\frac{1}{2}$ mo.
21. £764 8s. 7d.;	,, $6\frac{1}{2}$	,,	,,	,,	1 yr. 64 da.

22. A sum of £10 lay in a savings bank for 4 months, where the interest at 3 per cent. per ann. is calculated monthly and added to the principal. What total sum of interest would be due?

23. What will £13 amount to in $1\frac{1}{2}$ year if deposited in a bank where interest at 3 per cent. per ann. is calculated quarterly and added to the principal?

24. A sum of £3400 lies at simple interest for 2 years at the rate of 4 per cent. per ann. How much more would have been gained if it had been placed at compound interest, the calculations being made yearly?

25. In a certain bank the interest is calculated and added to the principal half-yearly, viz., on 30th June and 31st Dec. What interest would be due on 31st Dec., 1875, on a sum of £30, which had been deposited on 3rd April of the same year?

26. At the birth of a son the sum of a thousand pounds is invested for him until his coming of age, the interest to be the rate of 5 per cent. per ann. and added yearly to the principal. What will it have increased to by that time?

27. A money-dealer borrows £5000 for a year, the interest agreed upon being at the rate of 4 per cent. per ann. He lends the sum to a third person and receives interest at the rate of 1 per cent. per quarter, which as it falls due he lends out at the same rate. What profit does he thus realise?

28. A person with an annual income of £1500 puts out a tenth of it at compound interest on the same day every year. How much money will he thus have lying in his name on going to make his fourth investment, if the rate of interest be 4 per cent. per ann.?

29. A gentleman obtains a situation at an annual salary of £800, paid quarterly, with a rise of 50 per cent. yearly. If he save the last

quarter's salary of each year, and invest it in a bank where the interest at the rate of 4 per cent. per ann. is added yearly to the principal, what will he thus be possessor of 4 years after entering on his duties?

30. A person placed £6000 in a bank where interest at the rate of 3 per cent. per ann. was added half-yearly to the principal, but at the time of the addition of the interest he regularly withdrew £500. What sum would remain in his name two years afterwards?

In connection with the subject of compound interest, questions may arise similar to those considered on pp. 227—230, under the head of simple interest. They are not, however, so easy of solution; but at the same time they are of less practical importance, with the exception of those coming under the third head, viz., where the principal is to be found. To this head the examples following mainly belong.

Example 1. What sum placed in a bank, and having interest at the rate of 5 per cent. per annum added to it yearly, will increase to £1041 17s. 3d. in 3 years?

Under the same circumstances

£1 would increase to £$(1.05)^3$

i.e., " " £1.157625,

so that every £1 in the investment corresponds to a sum of £1.157625 in the amount to which the investment increased, viz., £1041.8625; hence, the number of times that 1.157625 is contained in 1041.8625 is the number of pounds in the investment.

$$\therefore \text{ sum required} = £1041.8625 \div 1.157625$$
$$= £900.$$

Or, more shortly, thus:—

We know that

$$\text{principal} \times (1.05)^3 = £1041.8625$$
$$\textit{i.e.}, \text{ principal} \times 1.157625 = £1041.8625$$
$$\therefore \text{ principal} = £1041.8625 \div 1.157625$$
$$= £900.$$

Example 2. A certain sum is placed at interest at the rate of 3 per cent. per annum, the interest being added yearly to the principal, and 4 years afterwards it has been increased by £250. Find the sum invested.

Under the same circumstances

$$£1 \text{ would increase to } £(1.03)^4$$
$$\textit{i.e.,} \quad ,, \quad ,, \quad £1.12550881$$
$$\text{and } \therefore £1 \text{ would be increased by } £\ .12550881,$$

so that the number of times that .12550881 is contained in 250 is the number of pounds in the investment.

$$\therefore \text{ sum required} = £250 \div .12550881$$
$$= £1991.892\ldots$$
$$= £1991 \ 17s. \ 10d. \text{ nearly.}$$

The kind of difficulty which arises in the examples coming under the other heads is seen in the following, and is worthy of attention.

Example 3. The sum of £12000 lay in a bank for 7 years, the interest being added to it yearly. Knowing that in that time it increased to £15791 6s., find the rate of interest per cent. per annum.

Here we know (p. 235) that

$$£12000 \times (\text{an unknown number})^7 = £15791.3,$$

the unknown number being, of course, 1 + the percentage on it,

$$\therefore (\text{the unknown number})^7 = £15791.3 \div £12000$$
$$= £1.3159\ldots.$$

But at this stage the learner has no means of finding the number which when raised to the seventh power would produce 1.3159..., so that he cannot proceed farther. With a little additional knowledge the number is found to be 1.04, and ∴ the rate per cent. per annum must have been 4.

Exercises. Set CII.

1. If interest be at the rate of 4 per cent. per ann. and be added yearly to the principal, what sum would by investment be increased in 3 years to £1124.864?

2. What sum of money would be increased in 4 years to £225101.762, if interest be added yearly to it at the rate of 3 per cent. per ann.?

3. A sum of money is placed in a bank where interest at the rate of 3 per cent. per ann. is added yearly to the principal, and in 2 years the total interest which has accrued amounts to £28.14. Find the sum invested.

4. A bank offers to depositors compound interest at the rate of 5 per cent. per ann. calculated yearly. What sum must now be invested in it to be worth £17624 16s. 9¾d. 4 years hence?

5. A sum of money was invested and three years later was found to have increased by £129.883. Knowing that interest at the rate of 5 per cent. per ann. had been added yearly to the principal, calculate the sum invested.

6. In negotiating for the purchase of a house a person offers £2000 ready money, or £2100 three years afterwards. If the seller can invest his money in such a bank as is mentioned in Exercise 3, which is the better offer, and why?

7. Find what capital will have increased in a year to £1446 3s. 1d., if entrusted to a money dealer who gives compound interest at the rate of 7 per cent. per ann. calculated half-yearly.

8. If in a bank interest at the rate of 3 per cent. per ann. be added quarterly to the investments, and on a certain deposit £200 of interest have accrued in a year, what was the amount of this deposit?

9. A certain sum was placed for 3 years at compound interest calculated yearly at the rate of 2½ per cent. per ann. Had it been put out at simple interest for the same time at the same rate £7 14s. 6d. less would have been realised. Find the sum.

10. A person invests a sum in a bank where interest at the rate of 3½ per cent. is added yearly to the deposits; at the same time he invests a like sum in another bank where the rate of interest is ½ per cent. less, but where the additions of interest are made half-yearly; and two years afterwards he finds that there is £7 of difference in the two accounts. Find the sum invested in both banks.

11. A certain sum deposited in the bank referred to in Exercise 8 would in a year gain 10s. more of interest than it would if deposited in the bank of Exercise 3. What is the sum?

12. The sum of £350 is deposited in a bank such as that of Exercise 4. How many years must elapse before it will have increased to £405 3s. 4½d.?

156. Discount.—If a sum of money be due at a future time, and it is desired to discharge the debt now, a smaller sum is usually taken in payment. This smaller sum is called the *present worth* or *present value* of the sum due at the future date; and the difference between the two sums is known as *discount*, which has been already referred to as an allowance or abatement made in consideration of the payment of a sum of money before it is due. Thus if a debt of £408 falls to be paid two years hence, and the creditor (that is, the person to whom it is due) accepts at

the present time £400 from the debtor as full payment, then £400 is called the *present value* of the debt mentioned, and the £8 of difference is called the discount on the debt.

The principle which ought to guide us in estimating what the present value should be, or what discount ought to be allowed in any case, is made evident if we consider the ground on which the creditor accepts with satisfaction a sum less than the total amount which is to fall due. This he does because he can place at interest the money he at present receives with the certain knowledge that it will amount to a larger sum by the time of the falling due of the debt. If, therefore, he accepts at present exactly such a sum as would, if laid out at interest either by himself or the debtor, amount to the actual debt at the future date referred to, he is clearly not a loser, the debtor is not a gainer, and the transaction is perfectly just.

It is thus seen that the present worth of a sum due at a future time and the discount on such a sum are dependent on the rate of interest obtainable; and that when this rate, the amount of the debt, and the date at which it is due are given, we may find (1) the *present worth* by calculating (as at p. 229) what sum now placed at interest at the given rate would amount to the given debt at the date specified, and (2) the *discount*, by taking the difference between the actual debt and its present worth, or by calculating the interest which if gained by a sum invested at the given rate for the time referred to would increase the sum to the given debt. For example, the rate of simple interest being 3 per cent. per annum,

if we invest £100 at the present date,
it will amount to £103 this day next year,
and to £106 this day the year after next.

Consequently with this rate of interest we know that in the case of a debt of £106 due two years hence

the present worth is £100,
the worth this day next year is £103,
the present discount is £6,
and the discount this day next year is £3.

For the sake of easy remembrance the essential statements in the above may be conveniently abbreviated and formulated as follows :—

Present Worth + Int. on Pres. Worth = Future Debt
Discount on Fut. Debt = Fut. Debt − Pres. Worth
= Int. on Pres. Worth.

Example 1. Calculate the present value of £545 due 4 yr. 2 mo. hence, the rate of interest being $3\frac{1}{4}$ per cent. per annum. Find also the present discount.

$$\begin{aligned}\text{Interest on £100 for } 4\tfrac{1}{6}\text{ yr.} &= £3\tfrac{1}{4}\times 4\tfrac{1}{6}\\ &= £\frac{13}{4}\times\frac{25}{6}\\ &= £\frac{325}{24}.\end{aligned}$$

∴ in $4\frac{1}{6}$ yr. £100 would increase to £$100\frac{325}{24}$;
so that

$$\begin{aligned}\text{present value of £}100\tfrac{325}{24} &= £100\\ \therefore \quad ,, \qquad £545 &= £100\times\frac{545}{100\frac{325}{24}}\\ &= £100\times 545\times\frac{24}{2725}\\ &= £100\times 1\times\frac{24}{5}\\ &= £480.\end{aligned}$$

$$\begin{aligned}\text{Hence, discount required} &= £545 - £480\\ &= £65.\end{aligned}$$

Example 2. What is the present discount on a sum of £3045 16s. 8d. due $2\frac{1}{2}$ yr. hence, the rate of interest being 3 per cent. per annum?

Here we might proceed as in Example 1, first calculating

the present value, and then deriving the discount; but the discount may also be calculated first.

$$\text{Interest on } \pounds 100 \text{ for } 2\tfrac{1}{2} \text{ yr.} = \pounds 3 \times 2\tfrac{1}{2}$$
$$= \pounds 7\tfrac{1}{2}.$$

$\therefore$ in $2\frac{1}{2}$ yr. £100 would become £$107\frac{1}{2}$;
so that

$$\text{discount on } \pounds 107\tfrac{1}{2} = \pounds 7\tfrac{1}{2}$$
$$\therefore \quad \text{,,} \quad \pounds 3045 \text{ 16s. 8d.} = \pounds 7\tfrac{1}{2} \times \frac{3045\frac{5}{6}}{107\frac{1}{2}}$$
$$= \pounds \frac{15}{2} \times \frac{18275}{6} \times \frac{2}{215}$$
$$= \pounds \frac{5}{1} \times \frac{3655}{2} \times \frac{1}{43}$$
$$= \pounds \frac{5}{1} \times \frac{85}{2}$$
$$= \pounds 212 \text{ 10s.}$$

If the rate of compound interest were given, the first portion of exercises like the preceding, viz., the calculation of what £100 would increase to in the given time, would necessarily be different. We might equally well, of course, find what £1 would increase to, and this is what is done in the solution of what may be looked on as a question of this kind, viz., Example 1, p. 235.

Example 3. A debt of £212 13s. 4d. is due on 17th October, but is discharged on 1st March. The rate of interest being $2\frac{1}{2}$ per cent. per ann., what discount should be allowed, and what is the value of the debt at the latter date?

$$\text{No. of days from 1st March to 17th October}$$
$$= 30 + 30 + 31 + 30 + 31 + 31 + 30 + 17$$
$$= 230.$$

$$\text{Interest on } \pounds 100 \text{ for 230 days} = \pounds 2\tfrac{1}{2} \times \frac{230}{365}$$
$$= \pounds \frac{5}{2} \times \frac{46}{73}$$
$$= \pounds \frac{115}{73}$$

$$\therefore \text{ discount on } \pounds 100\tfrac{115}{73} = \pounds \frac{115}{73}$$

$$\therefore \text{ discount on } £212 \text{ 13s. 4d.} = £\frac{115}{73} \times \frac{212\frac{2}{3}}{100\frac{115}{73}}$$
$$= £\frac{115}{73} \times \frac{638}{3} \times \frac{73}{7415}$$
$$= £\frac{23}{1} \times \frac{638}{3} \times \frac{1}{1483}$$
$$= £\frac{14674}{4449}$$
$$= £3 \text{ 5s. } 11\tfrac{867}{1483}\text{d.}$$

$$\therefore \text{ value of debt on 1st March} = £212 \text{ 13s. 4d.} - £3 \text{ 6s.}$$
$$= £209 \text{ 7s. 4d.}$$

The sums of money asserted in connection with these examples to be correct estimates of present value or discount, are not, however, the sums which would usually be taken as present value or discount if the examples were actual cases occurring in the ordinary course of business. Discount there appears chiefly in two different forms, viz., (I.) in connection with bills of exchange and promissory notes, (II.) in connection with the ordinary accounts rendered by tradespeople and merchants.

(I.) In illustration of the former, suppose a person has contracted a debt, and that instead of discharging it at once he finds it will be more convenient to make payment at a later date. If the creditor be willing, the debtor may make an engagement in the form of a promise to pay the debt or a sum in lieu of it at a certain length of time afterwards, say 3 months. The written instrument in which the debtor makes this promise, and which he gives to his creditor, is called a *promissory note.* Again, suppose that instead of paying the money himself he has a friend, who from being indebted to him, or from some other cause, is willing to pay the necessary sum at the future date, he may order this friend to do so. The written instrument, handed to his creditor, in which he gives this order, and on which his friend has to write his acceptance of the obligation, is called a *bill of exchange.* It is spoken of as a bill *at* 3 months, or is said to *run* for 3 months; and if the order be made, on 31st August say, it is said to be *drawn* on that day. Calen-

dar months being counted, it is nominally due on 30th November, but is not legally due until 3rd December, three days, called *days of grace*, being allowed. A bill of exchange may also serve exactly the same purpose as a promissory note, the order in this case being made by the creditor and accepted by the debtor, and no third person therefore being concerned.

If the holder of the note or bill presents it for payment on 3rd December, he ought of course to receive the full sum mentioned on it. But if he desires to be paid before this date, the same cannot be expected, as the person from whom he asks payment is entitled to discount. Usually it is a banker or "bill discounter" who is asked in such circumstances, and discount with them is a charge calculated, exactly like interest, at so much per cent. per ann. on the full debt for the length of time the note or bill has still to run. In making this deduction, and in paying the balance, the banker or other person is said to *discount the bill*, and the rate at which he calculates his charge is called the *rate of discount*.

Example 4. A bill for £540 12s. was drawn on 30th August, 1875, at 8 months, and discounted on 11th December, 1875, at $2\frac{1}{2}$ per cent. per ann. Calculate the discounter's charge and the sum paid to the holder of the bill.

The bill was discounted on 11th Dec.
and was not due till 3rd May,
∴ no. of days for which discount must be allowed

$$=20+31+29+31+30+3$$
$$=144.$$

Hence,

$$\text{bill discounter's charge} = £2\tfrac{1}{2} \times \frac{540.6}{100} \times \frac{144}{365}$$
$$= £5{\cdot}406 \times \frac{72}{73}$$
$$= £389{\cdot}232 \div 73$$
$$= £5 \text{ 6s. 8d. nearly.}$$

$$\text{And sum paid to holder} = £540 \text{ 12s.} - £5 \text{ 6s. 8d.}$$
$$= £535 \text{ 5s. 4d.}$$

If the rate of discount taken be the same as the rate of interest obtainable, the bill discounter makes a profit; for his charge is then the same as the interest on the *full debt* for the length of time the bill has to run, whereas the true discount is the interest on the *present worth* for the same time. This profit may be considered a return for the accommodation afforded to the holder; and being the difference between the interest on the full debt and the interest on the present worth, it must be the interest on the difference between the full debt and the present worth, that is, it must be the interest on the true discount.

Example 5. Calculate the discount which a bill discounter would charge on a sum of £3045 16s. 8d., due $2\frac{1}{2}$ years hence, the rate of discount being 3 per cent. per ann.

$$\begin{aligned}\text{Charge required} &= £3 \times \frac{3045\frac{5}{6}}{100} \times \frac{2\frac{1}{2}}{1}\\ &= £91{\cdot}375 \times \frac{5}{2}\\ &= £228.4375\\ &= £228\ 8\text{s. } 9\text{d.}\end{aligned}$$

The learner should carefully note the difference between this example and Example 2.

(II.) The discount connected with ordinary accounts is quite distinct from either of the forms already described, and is much less fixed in character. It is a percentage abatement made by tradesmen and merchants who give credit, in consideration of early payments, but it is only in a very loose way that the element of time enters into it.

For example, a tradesman announces his terms as "3 *months' credit: discount at* 5 *per cent. for ready money;*" or, it may be, "6 *months' credit:* 4 *per cent. of discount for payment within the first* 3 *months;*" the rate of discount being thus no longer so much per cent. *per ann.*, but simply so much per cent. A person who sells for ready money fixes his prices so as to gain a certain rate of profit. If, selling at the same prices, he ceases to insist on ready

money, his rate of profit must decrease, and to maintain it he puts an increase on his charges. In the event, therefore, of his being still offered ready money, he can afford to dispense with this additional charge, which he does by giving the so-called discount.

Exercises. Set CIII.

Find the present value of, and the true discount, on—

1.	£616 19s. 3d.	due	2 yr.	hence;	rate of interest	5	per cent.	per ann.
2.	£989	,,	2½	,,	,,	3	,,	,,
3.	£1246 1s.	,,	3¼	,,	,,	2	,,	,,
4.	£1334 8s.	,,	2	,,	,,	2⅛	,,	,,
5.	£500 3s. 6d.	,,	4	,,	,,	2	,,	,,
6.	£145 4s.	,,	2½	,,	,,	4	,,	,,
7.	£725 2s. 8d.	,,	3	,,	,,	2⅐	,,	,,
8.	£2279 9s. 4d.	,,	2	,,	,,	3½	,,	,,
9.	£1753 2s. 6d.	,,	1¼	,,	,,	2½	,,	,,
10.	£846 17s. 9d.	,,	160 da.	,,	,,	3⅛	,,	,,

Find the true discount (without first calculating the present value) on—

11.	£4676 18s. 3d.	due	3½ yr.	hence;	rate of interest	3	per cent.	per ann.
12.	£1333 1s. 9½d.	,,	2	,,	,,	3¾	,,	,,
13.	£1557 5s. 9d.	,,	1¼	,,	,,	3	,,	,,
14.	£524 6s 10½d.	,,	2½	,,	,,	2½	,,	,,
15.	£2251 13s. 9d.	,,	5	,,	,,	2½	,,	,,
16.	£672 13s. 11d.	,,	4	,,	,,	2½	,,	,,
17.	£830 10s. 1½d.	,,	7½ mo.	,,	,,	4	,,	,,
18.	£571 15s. 7½d.	,,	7½ yr.	,,	,,	2½	,,	,,
19.	£1234 6s. 10½d.	,,	1 yr. 10½ mo.	,,	,,	4	,,	,,
20.	£1085 1s. 6¾d.	,,	5 yr.	,,	,,	3¾	,,	,,
21.	£484 3s. 6d.	,,	164 da.	,,	,,	4½	,,	,,
22.	£969 10s. 9d.	,,	210 da.	,,	,,	3⅞	,,	,,

Find the present value and discount in the following cases, where *compound* interest calculated yearly is obtainable :—

23.	£694 11s. 6d.	due	3 yr.	hence;	rate of interest	5	per cent.	per ann.
24.	£946 8s.	,,	2	,,	,,	4	,,	,,
25.	£762 12s. 6d.	,,	2½	,,	,,	3	,,	,,
26.	£880 10s. 8d.	,,	400 da.	,,	,,	2½	,,	,,

27. A debt of £2243 8s., which was not due till Christmas, was discharged on 30th June. What discount should have been allowed, the rate of interest being 5 per cent. per ann. ?

28. A debt of £1888 falls to be paid on 1st Dec. What sum would

discharge the debt on 24th March of the same year, interest being allowed at 5 per cent. per ann.?

29. Calculate the value on 4th April of a debt of £1518 4s. which is not due till 20th Jan. of the following year, the rate of interest allowed being at 5 per cent. per ann.

30. If the rate of interest be 4 per cent. per ann., what discount should be allowed on a debt of £2688, which was discharged on 8th May and was not due till 20th July?

31. The sum of £752 12s. was due on 10th Oct., 1876. Find its value on 27th Feb., 1876, the rate of interest being 5 per cent. per ann.

32. What discount should be allowed when a debt of £400 10s. 6d. due on 1st July is discharged on 29th Feb., the rate of interest being 2½ per cent. per ann.?

33. The sum of £1122 18s. was due on 5th Aug., 1875. What would be its value on 31st Jan., 1875, interest being allowed at 5 per cent. per ann.?

34. On 21st May, 1876, a bill for £800 was drawn at 3 months, and on 20th June it was discounted at 8 per cent. per ann. Find the discounter's charge.

35. Goods to the value of £141 7s. 6d. are purchased on 7th March, and fall to be paid on 30th Nov. If the rate of interest be 3½ per cent. per ann., what would be the ready-money payment?

36. A bill discounter on 2nd July charges at the rate of 10 per cent. per ann. in the case of a bill for £600 drawn on 31st May at 6 months. What sum does the holder receive?

37. Find the present value of £35322 2s. 6d. due 2 years hence, the obtainable rate of compound interest calculated half-yearly being 5 per cent. per ann.

38. What would the holder of the following promissory note receive if it were discounted on 4th Aug. at the rate of 8 per cent. per ann.?

£400. *London, 18th April, 1876.*

Six months after date, I promise to pay Mr. James Brown or his order the sum of four hundred pounds.

William Benson.

39. If the rate of discount be 3 per cent. per ann., what sum will a person receive on 29th Feb. for a bill for £680 drawn on 1st Jan. at 6 months?

40. What would be the discounter's charge on 31st May in the case of the following bill, if the rate of discount were 4½ per cent. per ann.?

£1050. *Manchester, 4th May, 1875.*

Three months after date pay to Mr. John Johnson or his order the sum of one thousand and fifty pounds, value received.

David Williams.

To Mr. Joseph Watkins,
Tunbridge.

41. Calculate the discounter's charge in the case of a bill for

£240 10s. drawn on 4th March at 9 months and discounted on 27th April, the rate of discount being $3\frac{1}{2}$ per cent. per ann.

42. A legacy of £4000 falls to be paid to the legatee on his coming of age. What would be its value when he is 19 years old, the obtainable rate of compound interest calculated yearly being 3 per cent. per ann.?

43. On 30th Jan. a bill is drawn for £846 7s. 6d. at 3 months and is discounted on 29th Feb., the rate of discount being $6\frac{1}{2}$ per cent. per ann. If the rate of interest be $3\frac{1}{2}$ per cent. per ann. what gain does the bill discounter make?

44. A debt of £1000 due on 31st Dec., 1878, was discharged on 31st Dec., 1876. Assuming that compound interest calculated yearly at the rate of $3\frac{1}{2}$ per cent. per ann. was obtainable, find what discount should have been allowed.

45. The following bill was discounted on 18th Oct. at the then rate of interest, viz., 3 per cent. per ann. Find the bill discounter's profit.

£2600. *Glasgow, 8th July*, 1875.

Four months after date pay to me or to my order the sum of two thousand six hundred pounds, for value received.

Walter Campbell.

To Mr. Edgar Jeffreys,
Swansea.

46. A debt of £2000 due 3 years hence is discharged now, when the rate of interest obtainable is $3\frac{1}{2}$ per cent. per ann. How much more would the discount have been if this had been the rate of compound interest calculated yearly?

47. A person who has a debt of £250 due on 21st Aug. and a debt of £1000 due on 10th Dec. wishes to discharge them both on 30th June. What total sum should he pay, the rate of interest being 4 per cent. per ann.?

48. A workman who is paid monthly receives on pay-day £2 14s. 5d. less than his full wage, the difference being caused by his having received an advance a fortnight before. Knowing that in such a case the employer reckons as if the rate of interest were 12 per cent. per ann., find the sum which was advanced.

49. A tailor's terms are "4 months' credit; discount at 5 per cent. for cash." What is the ready-money payment for goods of the value of £40? What would it be if the rate of discount were 5 per cent. per ann., and what should it be if this were the rate of interest?

50. A person has a debt of £250 due 3 months hence, and another of £300 due 6 months hence. Supposing the rate of interest to be 3 per cent. per ann., find the total present value of the two, then calculate in what time the sum found would increase to £550 (*i.e.*, £250 + £300), and thus determine how many months hence £550 should be accepted in discharge of both debts.

51. A person has a debt of £400 due 6 months hence, and a debt of £600 due 9 months hence. Calculate how long hence the sum of the two full debts should be taken in discharge of both, the rate of interest being 5 per cent. per ann.

52. A merchant has debts of £600, £800, £900, due 3 months, 4 months, 6 months hence respectively. The rate of interest being $2\frac{1}{2}$ per cent. per ann., how many months hence ought he to pay £600 + £800 + £900 in full discharge of the separate debts?

53. Supposing that compound interest calculated yearly at the rate of 3 per cent. per ann. is obtainable, calculate what sum of money ought to be given on the last day of this year in return for an annuity of £100 payable on the same day of the next four years.

54. It is desired to purchase from a banker an annuity of £25 for a person whose age at present is nearly 67 years. What is the smallest sum the banker will ask for it if he expect that the annuitant will only reach three score years and ten, and if he can obtain compound interest calculated yearly at the rate of 4 per cent. per ann.?

157. STOCKS.—One way of investing money we have already referred to, and given examples of the calculations connected therewith, viz., placing it temporarily in the hands of a banker, or other person, who in return for the use of it gives *interest* at a specified rate. The learner's attention is now directed to two other modes of investment, viz., (I.) in connection with the capital of a *commercial company*, (II.) in connection with the debt of the Government of a State.

(I.) The promoters of a commercial company usually originate it publicly by the issue of a prospectus explaining the nature and object of the undertaking, specifying the capital required, the profit likely to be made, and the number of equal *shares* into which the capital is divided, and inviting the public to become *shareholders*. If sufficient applications be made, the shares are then allotted among the applicants, and part, at least, of the amount of each share having been paid, business is begun. Not unfrequently a share is £100, and part of it is paid on application, part on allotment, and other parts are called for at such times afterwards as may be thought best, until it is *fully paid up*. At the end of a year or half a year after starting, the profit, if any, is ascertained, and as much of it as may be deemed advisable is taken as a *dividend*, *i.e.*, a sum to be divided among the shareholders. If the prospects of the company be good, or money be plentiful, outsiders may wish to become shareholders; on

the other hand, if not, shareholders may desire to be so no longer: and thus there arises the need for the *sale* of shares at prices differing from the original value. If, for example, the dividend be at the rate of 10 per cent. per annum on the money invested, and there be a likelihood of it not falling lower, a person who is receiving elsewhere a return of only 4 or 5 per cent. for his money will gladly pay more than £100 for a hundred-pound share; indeed, it is possible he may be safe in paying £200 for it, for if the dividend be as he expects, at the rate of £10 per share, he is thus receiving 5 per cent. per annum on his outlay. Changes in the price of shares will be found taking place day by day, and even hour by hour, the fortunes of a commercial undertaking and the demand for money being usually in a state of greater or less fluctuation. The prices are regularly published, and information regarding sales, &c., is supplied daily by the newspapers, in the contracted and otherwise peculiar phraseology which has grown up with the business. A number alone is ordinarily used in specifying a price, *pounds* with us being the denomination which is understood. Thus, if a shareholder sells the shares he possesses for £102 10s. each, the shares of the company are said to be at 102½. Further, in so doing, he is said to *sell out* of the company; and if the price obtained in the transaction be published, the shares are said to be *quoted* at 102½, and 102½ is one of the *quotations* of the day. If shares be selling at their original or nominal value, they are said to be *at par;* if above it, they are said to be *at a premium*, or *above par;* if below it, *at a discount*, or *below par*. For example, if the shares of the company just referred to were hundred-pound shares, that is, if £100 were their nominal value, then when the above transaction took place they were 2½ above par, or at 2½ premium. Sometimes a portion of the capital of a company is raised on different conditions from the rest, viz., by having a fixed rate of dividend attached to it. The shares of such a portion,

in contradistinction to the *ordinary* shares, are known as *debenture*, *preference*, or *guaranteed* shares. Before shares have been fully paid up, it would be troublesome to deal in fractional parts of a share, and consequently only whole numbers of shares are sold; but afterwards this drawback in great part disappears, it being found almost as convenient that a person should own £325 or £476 of the capital of a company as that he should own £400, £500, or £900. It is, consequently, common to have arrangements made for this, and then *shares* are no longer spoken of, but *stock*, and any portion of the capital not less than £1 can be transferred from one person to another. Thus, 7 shares (if hundred-pound shares) would be represented by £700 of the stock or capital, or shortly, by £700 stock; 12 shares, if ten-pound shares, by £120 stock; and so on. Further, if the former were selling at 90, £700 stock would be worth £630, and it thus becomes necessary for the learner to distinguish carefully between a number of pounds *stock* and the same number of pounds *cash*.

(II.) The Government of a State, having incurred a debt which it finds itself for the time unable to pay, may borrow money for a limited period, very much as a private individual would. The mode in which our own Government commonly proceeds is to issue, in return for sums so advanced, bills (called "Exchequer Bills") containing an agreement to make repayment, with interest at the rate of so many pence per cent. per diem. These bills pass from hand to hand as money, and may be renewed from time to time, until the Government is in a position to pay them off. It very often happens, however, that a Government cannot entertain the hope of repaying a certain amount of such temporary loans, and what is then usually done is to resolve to hold this amount as a *perpetual* loan, at a certain fixed rate of interest. This operation is known as *funding*, and the debt is said to be no longer a *floating* but a *funded* debt. Of course, a loan may be funded from the first; that is to say, the money may

be advanced without recall from the Government, in return for perpetual annuities of so many pounds each. A person whom the Government credits with a portion of such a loan is called a *fund-holder;* and if the portion were, for example, £100, he would be said to possess £100 *Government stock*, or to have £100 in the funds. An acknowledgment of the debt is held by him, and his name stands on the register of fund-holders in connection with it, thus entitling him to be paid the interest when due. This acknowledgment the Government reserves to itself the right of, at any time, redeeming by payment of £100; whereas, the holder, if he wishes cash in return for his stock, or any part of it, cannot obtain it from the Government, but must find a purchaser in the stock-market, where the price to be obtained will be to his advantage or otherwise, according to circumstances. What he offers for sale may be viewed simply as the right to a perpetual annuity, the payment of which is guaranteed on the security of the Government, and this is always likely to be in demand as a safe investment. The funded debt of our own country, due in the main to engagement in foreign wars, amounts nearly to the enormous sum of £800,000,000. This affords a vast source of investment for the savings of the population; and the calculations connected with the subject of it are thus of great importance to the learner.

Government stocks are named from the fixed percentage attached to them, an additional designation being sometimes added for the sake of distinction. For example, 3 *per cent. stock*, 5 *per cent. stock*, *French* 3 *per cent. stock*, *India* 4 *per cent. stock;* or, more shortly, 3 *per cents.*, 5 *per cents.*, &c. The funds of the British Government are in great part 3 per cents., the largest portion being known as 3 *per cent. Consols*, from having originated by the *consolidation* of various loans into one uniform stock; and another as 3 *per cents. Reduced*, from having had originally a higher percentage attached to it.

The employment of the term *stock* to denote the capital required to redeem a funded debt is an extension of the use already explained. In the same way, the interest paid half-yearly to the fund-holders is, with still less accuracy, termed a *dividend*.

The purchase or sale of stock is usually made through a *stockbroker*, whose commission is a percentage (generally $\frac{1}{8}$) on the amount of stock dealt in, or on the proceeds in the case of certain railway and other stocks. Thus, Consols bought at $93\frac{3}{8}$ cost the person for whom the purchase is made $93\frac{1}{2}$, and bring to the seller only $93\frac{1}{4}$.

Example 1. If the 3 per cents. be at 95, what will be the cost of £1550 stock, and what the total expense of purchase, the brokerage being $\frac{1}{8}$ per cent.?

$$\begin{aligned}
\text{Cost of £100 Three per Cent. stock} &= £95 \\
\therefore \quad \text{,, \quad £1550 \quad ,, \quad ,,} &= £95 \times \frac{1550}{100} \\
&= £1472 \text{ 10s.} \\
\text{Brokerage} &= £\frac{1}{8} \times \frac{1550}{100} \\
&= £1 \text{ 18s. 9d.} \\
\therefore \text{ total expense of purchase} &= £1474 \text{ 8s. 9d.}
\end{aligned}$$

The last result may be got directly, thus :—

$$\begin{aligned}
\text{Total expense of £100 stock} &= £95 + £\tfrac{1}{8} \\
\therefore \quad \text{,, \quad £1550 \ ,,} &= £95\tfrac{1}{8} \times \frac{1550}{100} \\
&= £1474 \text{ 8s. 9d.}
\end{aligned}$$

Example 2. I invest the sum of £1235 18s. 9d. in the $3\frac{1}{2}$ per cents. when they are at $98\frac{7}{8}$. How much stock do I obtain?

$$\begin{aligned}
\text{Stock obtained for } £98\tfrac{7}{8} &= £100 \\
\therefore \quad \text{,, \quad £1235 18s. 9d.} &= £100 \times \frac{1235\frac{15}{16}}{98\frac{7}{8}} \\
&= £1250.
\end{aligned}$$

Example 3. What annual income is derived from £6500 Dutch 5 per cent. stock?

$$\text{Income from £100 stock} = £5$$
$$\therefore \quad ,, \quad £6500 \;,, \quad = £5 \times \frac{6500}{100}$$
$$= £325.$$

Example 4. When Consols are selling at $91\frac{1}{2}$, what sum will be realised from the sale of £2750 stock, the broker's charge being $\frac{1}{8}$ per cent.?

$$\text{Sum received for £100 stock} = £91\tfrac{1}{2} - £\tfrac{1}{8}$$
$$= £91\tfrac{3}{8}$$
$$\therefore \quad ,, \quad £2750 \;,, \quad = £91\tfrac{3}{8} \times \frac{2750}{100}$$
$$= £2512 \text{ 16s. 3d.}$$

Example 5. A person invests £2000 in the $3\frac{1}{2}$ per cents. when they are at 90. What yearly income will he thence receive?

Here we might, as in Example 2, find the amount of stock purchased, and then, as in Example 3, calculate the income required. But since every £90 invested purchases £100 stock, and therefore brings in $£3\frac{1}{2}$ of income, we may proceed more simply thus:—

$$\text{Income from £90 invested} = £3\tfrac{1}{2}$$
$$\therefore \quad ,, \quad £2000 \;,, \quad = £3\tfrac{1}{2} \times \frac{2000}{90}$$
$$= £77 \text{ 15s. } 6\tfrac{2}{3}\text{d.}$$

Example 6. When New Three per Cent. stock is at $92\frac{5}{8}$, how much of it must be sold out so as to realise £1018 17s. 6d.?

$$\text{Stock required to realise } £92\tfrac{5}{8} = £100$$
$$\therefore \quad ,, \quad ,, \quad £1018 \text{ 17s. 6d.} = £100 \times \frac{1018\frac{7}{8}}{92\frac{5}{8}}$$
$$= £1100.$$

Example 7. A sum of money is invested in Four per Cent. stock when it is selling at 78. What rate of interest per cent. per ann. is obtained on the money?

$$\text{Interest obtained per annum on } £78 = £4$$
$$\therefore \quad ,, \quad ,, \quad £100 = £4 \times \frac{100}{78}$$
$$= £5\tfrac{5}{39}.$$

Example 8. How much money must be invested in the Three per Cents. when they are at $91\frac{5}{8}$, so as to bring the investor an income of £600 a year?

$$\text{Sum invested to bring £3 a year} = £91\tfrac{5}{8}$$
$$\therefore \quad \text{,,} \quad \text{,,} \quad £600 \text{ ,,} = £91\tfrac{5}{8} \times \frac{600}{3}$$
$$= £18325.$$

Example 9. In which is it more advantageous to invest, the 3 per cents. at $91\frac{1}{4}$, or the $3\frac{1}{2}$ per cents. at 101?

Here we might find, as in Example 7, the rates of interest per cent. (or, better, per pound) per ann. that would be obtained in the two cases, and thus answer the question; or we may proceed as follows:—

$$\text{In the 1st case an expense of } £91\tfrac{1}{4} \text{ yields } £3$$
$$\therefore \quad \text{,,} \quad \text{,,} \quad £101 \text{ ,,} \quad £3 \times £\frac{101}{91\frac{1}{4}}$$
$$\textit{i.e.}, \; £3\tfrac{117}{365}$$

But in the 2nd case an expense of £101 yields £$3\frac{1}{2}$

$\therefore$ the second investment is the more advantageous.

Example 10. The sum of £1000 is invested in 4 per cent. stock when it is at $122\frac{1}{2}$, and sold out when it is at $112\frac{1}{2}$. Find what loss is thereby sustained, the brokerage being neglected.

Here we might find, as in Example 2, the amount of stock obtained for £1000, then, as in Example 4, the sum realised on selling out, and then subtract this sum from £1000; or we may proceed more easily as follows:—

$$\text{Loss sustained on } £122\tfrac{1}{2} = £122\tfrac{1}{2} - £112\tfrac{1}{2}$$
$$= £10,$$
$$\therefore \quad \text{,,} \quad £1000 = £10 \times \frac{1000}{122\frac{1}{2}}$$
$$= £81 \text{ 12s. } 7\tfrac{41}{49}\text{d.}$$

Example 11. If I transfer £21750 stock from the $3\frac{1}{2}$ per cents. to the 3 per cents. when the former is at $92\frac{1}{2}$ and the latter at 87, what amount of 3 per cent. stock do I obtain, and what is the consequent alteration in my income?

Here we might find, as in Example 4, the money value of the £21750 stock, and then calculate as in Example 2, how

much 3 per cent. stock would be got for this sum; but it is easier to proceed as follows:—

(1) Stock obtainable when the price is $92\frac{1}{2} = £21750$

$\therefore$ „ „ $87 = £21750 \times \frac{92\frac{1}{2}}{87}$

$= £23125.$

(2) Income from original stock $= £3\frac{1}{2} \times \frac{21750}{100}$

$= £761$ 5s.

and „ new „ $= £3 \times \frac{23125}{100}$

$= £693$ 15s.

$\therefore$ diminution of income $= £67$ 10s.

Exercises. Set CIV.

1. What annual income is derived from £1560 of $3\frac{1}{2}$ per cent. stock?

2. What amount of stock can be bought at 94 for £4230?

3. A person owns £4555 $3\frac{1}{2}$ per cent. stock. Find his annual income from it.

4. When the $3\frac{1}{2}$ per cents. are at $96\frac{7}{8}$, how much money shall I receive for £6480 of this stock?

5. Find the cost of £7500 Bank stock at $221\frac{1}{2}$, the broker's charge being taken into account.

6. How much stock can be bought at $92\frac{1}{2}$ with £22230, brokerage being taken into account?

7. What money shall I receive for £6550 4 per cent. stock selling at $106\frac{5}{8}$, brokerage at the rate of $\frac{1}{8}$ per cent. being paid?

8. When 3 per cent. stock is selling at $89\frac{5}{8}$ the sum of £7000 is invested in it. How much stock is obtained?

9. The possessor of £3460 4 per cent. stock sells out at 105. What sum does he realise, the broker's charge being taken into account?

10. What half-yearly dividend will be derived from £1845 $3\frac{1}{2}$ per cent. stock?

11. When 3 per cent. stock is at $92\frac{1}{8}$ how much must be sold out to pay a debt of £506, brokerage being taken into account?

12. What annual income will be derived from the investment of £3660 in Russian 5 per cents. when they are at $91\frac{1}{2}$?

13. 3 per cent. stock is purchased at $92\frac{3}{8}$. What rate of interest per cent. per ann. does the purchaser receive for his money, the sum paid for brokerage being considered?

14. The sum of £4500 is invested in the 4 per cents. when they are selling at 106. What annual income will the investor thereby receive?

15. If £6300 be invested in Bank stock at 252, how much stock is obtained and what is the broker's charge?

16. A person bought through a broker £1950 stock which cost him £1735 10s. What was the selling price?

17. Which is the better investment, the 3 per cents. at $91\frac{3}{8}$, or the $3\frac{1}{2}$ per cents. at $106\frac{5}{8}$, brokerage being taken into account?

18. Find what annual income a fund-holder will derive who possesses £6500 3 per cent. stock and £8450 $3\frac{1}{2}$ per cent. stock?

19. If the 3 per cents. be at $91\frac{3}{4}$ and the 4 per cents. at $106\frac{3}{4}$, which will afford to the investor the higher rate of interest?

20. Calculate the cost of £5500 3 per cent. stock and £4600 5 per cent. stock, when the former is at £$7\frac{1}{2}$ discount and the latter at £$6\frac{3}{4}$ premium.

21. If £5200 be entrusted to a broker for the making of an investment in the French 5 per cents. at $103\frac{7}{8}$ and the payment of his commission thereon, what amount of stock will be obtained?

22. A person desires an annuity of £100. What will be the expense of obtaining it, including brokerage, if 3 per cent. Consols be purchased at $93\frac{1}{2}$?

23. The cost of £7500 5 per cent. stock, including brokerage, is £8000. How much is the price of stock above par?

24. On the shares of a trading company £4 10s. has been paid, and the half-yearly dividend is 15s. per share. What rate of interest per cent. per ann. are the shareholders receiving for their money?

25. The annual sum paid in dividends to holders of 3 per cent. Consols is about £11856000. How much of this stock is there?

26. A father leaves £16082 in equal portions to two sons: the one invests his legacy in the 3 per cents. at $93\frac{1}{2}$, the other in the 4 per cents. at $107\frac{1}{2}$. Find the difference of their half-yearly dividends.

27. A person who owns £4000 3 per cent. stock sells out as much as will pay a debt of £423. The 3 per cents. being at 94, calculate how much stock still remains in his name.

28. What is the amount of stock acquired by a person who transfers £6220 stock from Hungarian 5 per cents. at $56\frac{3}{4}$ to the 6 per cents. at $77\frac{3}{4}$?

29. £18750 is invested in the 3 per cents. at $93\frac{1}{2}$. Find the investor's net income therefrom, after an income tax of 4d. in the pound has been deducted.

30. A speculator who before the deposition of the Sultan in 1876 bought Turkish 6 per cents. at $12\frac{7}{8}$, sold them afterwards at $15\frac{1}{2}$. Find his gain per cent.

31. £6900 is invested in Consols at 92. What loss does the investor sustain by a fall of $1\frac{1}{2}$ in the funds?

32. An investor who had 50 £10 shares allotted to him in an oil company, sold them out when they were quoted at £22 premium, and invested the money in India 4 per cents. at par. What would be his yearly income from this stock?

33. A person buys £7200 India 4 per cent. stock at $101\frac{1}{2}$ and sells it out at $102\frac{1}{4}$. Find the amount of his gain, taking into account the brokerage in both transactions.

34. A fund-holder transferred £6500 stock from the 3 per cent. Consols at 93¾ to the India 5 per cents. at 104¼. Calculate the income he would derive from the latter.

35. £42000 stock is transferred from the 3 per cents. at 92¾ to the 4 per cents. at 106¼. What alteration will thus be made in the holder's half-yearly dividend ?

36. The 3 per cents. Reduced amount to about £105000000 stock. What sum would be required to redeem this portion of the National Debt at its present price, viz., 93½ ?

37. The profits of a public company whose capital consists of £2000000 ordinary stock and £200000 6 per cent. preference stock amounted in one year to £70000. If the whole profits had been divided among the stockholders, what rate of dividend would have been declared on the ordinary stock ?

38. A person buys £4500 3 per cent. stock at 95⅝, sells out a third of it at 94⅛, £1600 at 96⅝ and the remainder at ⅛ premium. Find what sum he thus gains, the broker's commission being taken into account.

39. A fund-holder sells a quantity of 4 per cent. stock at 102½, and realises £1125, which he invests in another 4 per cent. stock at 98. How is his income improved by the transference ?

40. £3294 is invested in Government stock at 91½, and soon after there being a rise of ⅛, half the stock is sold out, and the other half when there is a further rise of ⅛. What gain has the investor made ?

41. £6000 4 per cent. stock is sold out at £2 discount, and the proceeds are invested in a mercantile company's £25 shares at par, which pay a dividend of £1 10s. per share. Find the improvement in the investor's income.

42. If £9350 be invested in the New 3 per cents. at 93¼, and the stock afterwards sold at 93¾, what expense is incurred for brokerage ?

43. A person who possesses £3000 invests the half of it in the 3 per cents. at 92½, and the other half in the 4 per cents. when they are at 1¼ premium. Find the average rate of interest received for his money.

44. If I invest in the 4 per cents. and receive interest at the rate of 5 per cent. per ann. for my money, how much is the stock below par ?

45. On 31st March, 1855, the funded debt of the British Government was made up as follows:—£3007775 2½ per cents., £745333404 3 per cents. (viz., Consols, Reduced and New), £2871515 3½ per cents., and £433124 5 per cents. Find how much of the annual national expenditure was due to dividends on debt.

46. A person invests £5187 10s. in the 3 per cents. at 83, and when they have risen to 84 he transfers three-fifths of his capital to the 4 per cents. at 95. Find the alteration in his income.

47. A person invests £5460 in the 3 per cents. at 91; he sells out £2000 stock when they have risen to 93½, and the remainder when they have fallen to 85; he then invests the produce in 4½ per cents. at 102. What is the difference in his income ?

48. A public company is paying a yearly dividend of £6 per share, and the shareholders are receiving the same rate of interest for their

money as if it had been invested in a $3\frac{1}{2}$ per cent. stock at 98. Find the market value of a share.

49. A person on the day that he was paid his half-yearly dividend from the 3 per cents. sold out at $89\frac{3}{4}$, the total sum received, after the deduction of brokerage, being £3645. How much of this was dividend, and what amount of stock had he possessed?

50. A person has an income of £350 a year from money in the 3 per cents. and 4 per cents. Knowing that he owns £6000 of the former stock, find how much of the latter must stand in his name.

51. The 3 per cents. are at $91\frac{1}{2}$, and the 5 per cents. are an equally good investment. At how much premium does the latter stock stand?

52. When the India 4 per cents. are at 102 a stockholder sells out, and when they fall to $100\frac{1}{8}$ buys in again. Knowing that he now possesses £4000 stock, find by how much he has improved his income.

53. A person who buys into the 3 per cents. finds that after paying an income tax of 4d. in the pound his net income is at the rate of 3 per cent. for the money invested. At what price was the stock purchased?

54. "If the 3 per cents. be at 95 and the Government offer to receive tenders for a loan of £5000000, the lender to receive five millions in the 3 per cents. together with a certain sum in the $3\frac{1}{2}$ per cents., what sum in the $3\frac{1}{2}$ per cents. ought the lender to accept?"

55. A speculator invested in India 4 per cent. stock at $101\frac{1}{4}$, and in selling out made a gain of 4 per cent. How much had the stock risen in price?

56. "A person sells Midland Railway stock, paying $6\frac{1}{2}$ per cent., at $128\frac{1}{2}$, and invests in Great Western Railway stock, paying 3 per cent., at $72\frac{1}{2}$. By how much per cent. per ann. will the interest received for his money be altered?"

57. £13800 is invested in $3\frac{1}{2}$ per cents. at 92, and the dividends as paid are invested in the same stock, the price being $\frac{1}{2}$ lower at each successive investment. Calculate the third dividend received.

58. The dividends on the 3 per cent. Consols are paid on 5th Jan. and 5th July, the dividends on the New 3 per cents. are paid on 5th April and 5th Oct. What money must a person who already holds £1500 of the former stock and £2500 of the latter invest so as to secure an income of £400 a year paid quarterly, Consols being at 93 and the New 3 per cents. at $91\frac{1}{2}$?

59. When the 3 per cents. were at 90, I found that by selling out and investing in India 4 per cents. at 95, I could improve my income by £24 6s. What was the amount of my stock in the 3 per cents.?

60. A person invests a sum of money in the 3 per cents. at $92\frac{3}{4}$ and a like sum in the 4 per cents. at $123\frac{1}{4}$, and finds that his annual income from the one is 5s. more than from the other. Calculate the total sum invested.

158. INTERNATIONAL EXCHANGE.—If a person in a

foreign country, say France, receives from England a quantity of goods, for which he is charged £100, it is clear that he must disburse such an amount of French money as may be an equivalent of the sum in question, and that a calculation must be made to find this amount.

As a basis of calculation he might compare the French gold coins with ours as to purity and weight, and finding as he would do that 1 sovereign contains as much pure gold as a French gold piece of 25.17 francs, it would follow that he had to send to England 25.17 francs × 100, *i.e.*, 2517 francs. Or, again, he might begin by finding the amount of pure gold in a sovereign, and multiplying the result, viz., 7.316 grammes, by 100, he might resolve to send 731.6 grammes of pure *uncoined* gold. In the former case he would be said to remit *specie* in payment, in the latter to remit *bullion*.

Neither of these courses, however, would in all likelihood be followed. Just as a merchant in Glasgow who owed the sum would not send a package of sovereigns in payment, so the French merchant has a simpler way than those referred to, viz., by means of a *bill of exchange* (see p. 243).

The Glasgow merchant would go to a banker, pay him £100 and a small sum additional for the accommodation, and would receive an *order* (not called a bill of exchange, but a *draft*) on a London bank, which he would forward to his creditor, and the creditor on presenting it at the bank in London would receive £100 in return. The Glasgow banker undertakes to forward sums thus paid to him, but as the London bank may have issued similar orders on the Glasgow bank, no transfer of specie may be necessary between them. In like manner the French merchant would procure from a banker or other person a *bill of exchange* for £100, payable in London, and this would be forwarded and cashed as the draft would be. Here, too, no transfer of specie might be needed, as the debts due by France to England might be equal in amount to those due by England

to France. The *price* paid for the bill would be at the rate of so many francs per pound; but the important point to be noticed is that this number of francs would not necessarily be 25.17. In fact, the rate would vary from time to time; for should there be a preponderance of debt due by France to England, and in consequence a great demand for bills and a probable necessity for remitting specie, the French bankers would naturally charge higher for the accommodation afforded; while, if the opposite were the case, they would willingly take less, thus changing the price, as dealers in other commodities do, in accordance with the law of supply and demand. Bills are spoken of as bills *on* the place in which they are payable, and are sometimes called *paper* of that place. The rate of exchange, which depends solely upon the weight and purity of the coins of the two countries, is known as the *intrinsic* or *par* rate of exchange, or, shortly, the *par* of exchange between the two; and the rate at which exchanges are actually made, or *current* rate, is spoken of as the *course* of exchange. If the course of exchange between Paris and London were quoted at "25.17," we should therefore say that exchange between the two cities was at par; and if the quotations were above 25.17, we should say that the rate of exchange was *against* French merchants and *for* ours, or that British money was at a premium, and similarly in the other possible case. The current rate of exchange can never be much above or below par, for a merchant who found that it would cost more to pay a foreign debt by means of a bill of exchange than by the transmission of specie could adopt the latter method; and, besides, if the rate of exchange be against a country, its merchants would be less eager to import, and those of the other country more so, *i.e.*, there would be a double tendency to equalise the debt between the two countries, and so bring back the rate of exchange towards par.

The mode above explained of using a bill of exchange in the payment of a foreign debt is not, however, the only one. The French merchant, instead of remitting a bill on

London, might order his creditor to *draw* on him in France. In obedience to this, the creditor would cause a bill of exchange, payable by the debtor, to be drawn out, and would proceed to *negotiate* it, *i.e.*, to sell it or receive his money in return for it.

Further, various other modes might be followed. Instead of remitting to his creditor a bill on London, the debtor might remit a bill on Amsterdam, say, which would be negotiated as before; or he might give an order to draw on Amsterdam; or, still less directly, he might remit to a correspondent in Amsterdam, who would remit to another in Hamburg, whence a final remission would be made to London; and so on—the object to be gained by any such indirect mode being the payment of his debt of £100 with the smallest possible number of francs.

Finally, calculations of exchange are sometimes complicated by the currency of a country being depreciated; that is to say, by the substitutes in circulation for the standard coins being of less value than the standard coins themselves. Thus, in America, 100 gold dollars are at present worth considerably more than 100 of the paper dollars current; if the number be 118, gold is said to be at 18 per cent. premium, and this is what is indicated by such a quotation as "gold, 118." The rates of exchange, it must be noted, refer always to the standard coins.

Example 1. When the rate of exchange is 20.64 marks for £1, how many marks, &c., will be given for £540 16s. 8d.? and how many pounds, &c., for 5000 marks?

(1)
$$\begin{aligned} \pounds 1 &= 20.64 \text{ marks} \\ \therefore \pounds 540\ 16\text{s. } 8\text{d.} &= 20.64 \text{ marks} \times 540\tfrac{5}{6} \\ &= 11162.80 \text{ marks} \\ &= 11162 \text{ marks } 80 \text{ pfennige.} \end{aligned}$$

(2)
$$\begin{aligned} 20.64 \text{ marks} &= \pounds 1 \\ \therefore 5000 \text{ marks} &= \pounds 1 \times \frac{5000}{20.64} \\ &= \pounds 500000 \div 2064 \\ &= \pounds 242\ 4\text{s. } 11\tfrac{1}{2}\text{d. nearly.} \end{aligned}$$

Example 2. How many pounds sterling, &c., are equivalent to 6480 dollars of American currency when gold is at 114, and the rate of exchange is 4.88 dollars for £1?

114 dollars in currency = 100 dollars in gold,

$$\therefore 6480 \quad ,, \quad = \quad ,, \quad ,, \quad \times\frac{6480}{114}$$

$$= 5684.21 \text{ dollars in gold.}$$

But 4.88 dollars in gold = £1

$$\therefore 5684.21 \quad ,, \quad = £1 \times \frac{5684.21}{4.88}$$

$$= £568421 \div 488$$

$$= £1164\ 15s.\ 11\tfrac{1}{4}d. \text{ nearly.}$$

Or we may condense the reasoning as follows:—

$$\left.\begin{array}{l} \text{Required amount} = 6480 \text{ dollars in currency,} \\ 114 \text{ dollars in currency} = 100 \quad ,, \quad \text{in gold,} \\ \text{and } 4.88 \quad ,, \quad \text{gold} \quad = £1, \end{array}\right\}$$

$$\therefore \text{required amount} = £\frac{6480 \times 100 \times 1}{114 \times 4.88}$$

$$= £\frac{810 \times 10000}{114 \times 61}$$

$$= £\frac{1350000}{19 \times 61}$$

$$= £1164\ 15s.\ 11\tfrac{1}{4}d. \text{ nearly.}$$

Example 3. A London merchant wishes to pay a debt of 6400 francs to a Paris merchant, when the course of exchange is quoted at 25.40 in London, and 25.45 in Paris. What gain is there to the former merchant if, instead of remitting, he be drawn upon by the latter?

The latter draws for such a number of pounds as would in Paris produce 6400 francs. Now in Paris

25.45 francs are the produce of £1

$$\therefore 6400 \quad ,, \quad ,, \quad £1 \times \frac{6400}{25.45}$$

So that the *sum drawn for* = £640000 ÷ 2545

$$= £251\ 9s.\ 5\tfrac{1}{2}d. \text{ nearly.}$$

On the other hand, the former would have paid in London such a number of pounds as were there the equivalent of 6400 francs, and

$$\text{this no. of pounds} = £\frac{6400}{25.40} = £64000 \div 254$$
$$= £251\ 19\text{s.}\ 4\tfrac{1}{2}\text{d. nearly.}$$
$$\therefore \text{gain required} = 9\text{s.}\ 11\text{d.}$$

Example 4. A London jeweller, who has a debt of 23844.95 francs to pay in Geneva, remits bills on Amsterdam when the rate of exchange between London and Amsterdam is 11 florins $19\frac{1}{4}$ stivers for £1, and that between Amsterdam and Geneva $212\frac{3}{4}$ francs for 100 florins. What sum does he pay in London? (One florin = 20 stivers.)

$$\left.\begin{aligned} \text{Required sum} &= 23844.95 \text{ francs,} \\ 212\tfrac{3}{4} \text{ francs} &= 100 \text{ florins,} \\ \text{and } 11.9625 \text{ florins} &= £1. \end{aligned}\right\}$$

$$\therefore \text{required sum} = £\frac{23844.95 \times 100 \times 1}{212.75 \times 11.9625}$$
$$= £936\ 18\text{s.}\ 6\tfrac{1}{2}\text{d. nearly.}$$

Example 5. When the rate of exchange between London and Vienna is 14.05 florins for £1, between Vienna and Geneva $181\frac{1}{2}$ francs for 100 florins, and between Geneva and Paris $99\frac{7}{8}$ francs for 100 francs, what is the rate of exchange between London and Paris by way of Vienna and Geneva?

$$\left.\begin{aligned} \text{Required no. of francs} &= £1, \\ £1 &= 14.05 \text{ florins,} \\ 100 \text{ florins} &= 181.5 \text{ francs,} \\ \text{and } 99.875 \text{ francs} &= 100 \text{ francs.} \end{aligned}\right\}$$

$$\therefore \text{required no. of francs} = \frac{1 \times 14.05 \times 181.5 \times 100}{1 \times 100 \times 99.875}$$
$$= \frac{1405 \times 1815}{99875}$$
$$= 25.53\ldots$$

Exercises. Set CV.

Find the cost in London of a bill of exchange for—

1. 5500 francs on Paris at 25.35 (francs for £1).
2. 2020 rupees on Bombay at 1s. $9\frac{3}{4}$d. (for 1 rupee).
3. 460.50 dollars on New York at 4.88 (doll. for £1).
4. 410.45 roubles on St. Petersburg at $32\frac{1}{2}$ (d. for 1 rouble).
5. 3015 marks on Hamburg at 20.44 (marks for £1).

6. 700460 reis on Oporto at $52\frac{1}{4}$ (d. for 1 milreis, *i.e.*, 1000 reis).

7. 6140 gulden on Vienna at 14.20 (gulden for £1).

8. 345.25 guilders on Amsterdam at 11.95 (guild. for £1).

9. 670.30 pesetas on Cadiz at 25.64 (pes. for £1).

10. 864.75 lire on Genoa at 25.70 (lire for £1).

11. If it were necessary to remit £150 17s. 6d. to Paris, £200 16s. to Bombay, and £175 14s. 3d. to St. Petersburg, what bills on these places would be obtained at the foregoing rates?

12. If it were necessary to remit to London £214 12s. 6d. from New York, £500 14s. 10d. from Hamburg, and £75 5s. from Oporto, what would the bills cost at the rates given above?

13. What would the sale of a bill of exchange on London for £120 8s. 6d. produce in Vienna, Amsterdam, and Genoa at the rates already mentioned?

14. If the course of exchange between London and St. Petersburg be quoted at $32\frac{3}{4}$, how much English money will be got for 45 roubles 45 copeks? (100 copeks = 1 rouble).

15. How many pounds, &c., are obtainable for 1320 florins 60 kreutzers when the rate of exchange between London and Vienna is quoted at 14.12? (100 kreutzers = 1 florin).

16. What amount of American currency is equal to £500 14s. 6d. when gold is quoted at $115\frac{1}{2}$ and the course of exchange at 4.89?

17. If the cost of a bill on Paris for 8974 francs be £400 12s. 6d., what is the course of exchange?

18. A bill of exchange for £640 12s. 6d. at sight is negotiated in Vienna when the course of exchange is quoted at 14.15. What sum ought the buyer to pay?

19. How many pounds, &c., must be paid in London to settle a debt of 10000000 reis in Lisbon, if bills on Paris be bought and forwarded to Lisbon when the course of exchange between London and Paris is quoted at 25.35, and between Paris and Lisbon at 542 (francs for 100 milreis)?

20. The sum of £250 10s. is laid out in London on bills of exchange on Paris at the rate of 25.30 francs per £1, and the bills having been remitted to Vienna are negotiated there at the rate of 178 francs for 100 florins. What sum do they bring?

21. Had bills on Vienna at the rate of 14.15 been bought with the £250 10s. of Exercise 20, how much more or less would the holder in Vienna have received?

22. The par of exchange between London and Paris is 25.17, and the course of exchange is quoted at 25.42. At how much per cent. premium is English money?

23. When the course of exchange between London and New York is quoted at 4.96, London exchange (*i.e.*, English money) is said to be at 2 per cent. premium. Calculate from this the par of exchange.

24. A merchant in Paris has a debt due to him in Amsterdam, in payment of which he may draw on Amsterdam, or cause bills on Paris to

be remitted to him. Which is the preferable course, if in Paris the quotation is "Amsterdam, $212\frac{1}{2}$" (francs for 100 guilders), and in Amsterdam "Paris, 55" (guilders for 120 francs)?

25. An American sends to his son in this country a bill of exchange for £220 when gold is at 15 per cent. premium and the rate of exchange is 4.87. What would the bill cost in currency, there being a banker's charge of $\frac{1}{8}$ per cent.?

26. A merchant in Geneva who has a debt of £1150 to pay in London may remit paper on Paris at $100\frac{1}{8}$ which is quoted in London at 25.40, or paper on Amsterdam at $211\frac{1}{4}$ which is quoted in London at 11.95. Which method is the more advantageous, and by how much?

27. When gold is quoted in New York at 110 the cost of a bill on London for £240 17s. 6d. is 1293.02 dollars in currency. What is the current rate of exchange?

28. A sum of £220 is handed in at a bank to pay for a bill of exchange on Paris and the banker's charge of $\frac{1}{8}$ per cent. on the same. If the course of exchange be quoted at 25.25, what sum will the holder in Paris receive?

29. For how large a sum can a person with £450 obtain a bill of exchange on Hamburg, when the course of exchange is quoted at 20.45, and there is a banker's charge of $\frac{1}{10}$ per cent.?

30. The cost in New York of a bill for £648 12s. 6d. on London is \$3577.65 when the rate of exchange is quoted at 4.80. What must be the quotation for gold?

31. Messrs. A and B of Paris have to make a remittance of 40000 dollars to New York, and this is accomplished by their correspondent in England buying London paper on New York at 4.88, and drawing upon them in return at 22.45. What sum must their remittance thus cost them?

32. Mr. A of London remits £460 to Mr. B of Paris when the rate of exchange is quoted at 22.40; with the sum received Mr. B buys paper on Vienna at 180, and remits it to Mr. A, who negotiates it at 14.15. What is the result of these transactions?

33. A London merchant wishes to remit paper to Paris in payment of a debt. Which paper is best, as deduced from the following?

Quotations in London.		*Quotations in Paris.*	
Paris	25.40.		
St. Petersburg	$32\frac{3}{4}$.	Pétersbourg	350 [fr. for 100 r.]
Vienna	14.20.	Vienne	180 [fr. for 100 g.]
Genoa	$25.72\frac{1}{2}$.	Gênes	$99\frac{7}{8}$ [fr. for 100 fr.]

34. The American gold dollar weighs 1.672 gram, and is $\frac{9}{10}$ fine, the French twenty-franc piece weighs 6.451 grams, and is of the same fineness. Calculate the par of exchange between the two countries, stating it as so many francs per dollar.

35. A London exchange dealer wishes to procure paper on St. Petersburg, and finds that he can have it from Paris and from Geneva by remitting paper on these places in return. Deduce from the following quotations which is the better place to deal with:—

London Quotations.	*Paris Quotation.*	*Geneva Quotation.*
Paris 25.70.	Pétersbourg 349.	Pétersbourg 350.
Geneva 25.80.		

36. British standard gold is 22 carats fine (*i.e.*, 22 parts out of 24 are pure), and 40 lb. troy of it are coined into 1869 sovereigns. Find the par of exchange between France and England, giving it in francs per pound. (See Exercise 34.)

37. A London firm, a partner of which carries on a branch of the business in Paris, has a debt to pay in Vienna, and this may be accomplished in one of the following ways, viz. :—(1) by remitting London bills on Vienna to the creditor, (2) by remitting London bills on Paris to the partner, and by the latter remitting Paris bills on Vienna to the creditor, (3) by remitting London bills on Paris to the partner, and by ordering the creditor to draw on the partner in Paris, (4) by ordering the creditor to draw on the partner and the partner to draw on the London house. With the following quotations, which is the best way?

London Quotations.	*Paris Quotations.*	*Vienna Quotation.*
Vienna 14.20.	Vienne $180\frac{1}{8}$.	Paris $41\frac{1}{4}$ [for 100 fr.]
Paris 25.60.	Londres 25.16.	

159. Sharing of Profits, &c.—Here there are few technical terms requiring to be explained: the subject will be sufficiently understood from a study of the examples following, and the reasoning made use of in their solution.

Example 1. Three brothers have £10, £5, £1, respectively, the whole of which is put into a savings-bank as one sum, the total interest received being £2. How should this be divided among them?

$$\text{Interest on the whole, viz., } £16 = £2$$
$$\therefore \text{ interest on } £10, \textit{ i.e.}, \text{ 1st share} = £2 \times \frac{10}{16}$$
$$= £1\ 5\text{s.}$$
$$\text{interest on } £5, \textit{ i.e.}, \text{ 2nd share} = £2 \times \frac{5}{16}$$
$$= 12\text{s. } 6\text{d.}$$
$$\text{and interest on } £1, \textit{ i.e.}, \text{ 3rd share} = £2 \times \frac{1}{16}$$
$$= 2\text{s. } 6\text{d.}$$

Example 2. Three merchants enter into partnership, contributing respectively £1000, £1500, and £2000, for the carrying on of their business. After a certain time they find that they have lost £3000, and the partnership is dissolved. What share should each bear of this loss?

$$\text{Total capital} = £1000 + £1500 + £2000$$
$$= £4500.$$

Now, since the loss on £4500 = £3000;
hence,

$$\text{1st share, } i.e., \text{ loss on } £1000 = £3000 \times \frac{1000}{4500}$$
$$= £666\ 13s.\ 4d.$$
$$\text{2nd share, } i.e., \text{ loss on } £1500 = £3000 \times \frac{1500}{4500}$$
$$= £1000$$
$$\text{3rd share, } i.e., \text{ loss on } £2000 = £3000 \times \frac{2000}{4500}$$
$$= £1333\ 6s.\ 8d.$$

Example 3. Two partners advance £1000 each on starting business together: three months afterwards they admit two additional partners, one advancing £1200, the other £800, and at the end of the year the profits, amounting to £2100, are divided. Find the share of each partner.

1st partner	has	£1000	invested for	12 mo.
2nd	,,	£1000	,,	12 ,,
3rd	,,	£1200	,,	9 ,,
and 4th	,,	£ 800	,,	9 ,,

which is the same as if

1st partner	had	£12000	invested for	1 mo.
2nd	,,	£12000	,,	,,
3rd	,,	£10800	,,	,,
and 4th	,,	£ 7200	,,	,,

Now £12000 + £12000 + £10800 + £7200 = £42000;

$$\therefore \text{1st partner's share} = £2100 \times \frac{12000}{42000}$$
$$= £600.$$
$$\text{2nd} \quad ,, \quad = £600.$$
$$\text{3rd} \quad ,, \quad = £2100 \times \frac{10800}{42000}$$
$$= £540.$$
$$\text{4th} \quad ,, \quad = £2100 \times \frac{7200}{42000}$$
$$= £360.$$

Example 4. 120 oranges are divided among four boys, and for every 6 which the eldest receives, the second receives 3, the third 2, and the youngest 1. What are their shares?

$6+3+2+1=12$; so that out of every 12 given away, the eldest receives 6, *i.e.*, he receives $\frac{6}{12}$ of the whole; and so on. Hence,

$$\text{share of eldest}=120\times\frac{6}{12}=60$$
$$\text{,,} \quad 2\text{nd}=120\times\frac{3}{12}=30$$
$$\text{,,} \quad 3\text{rd}=120\times\frac{2}{12}=20$$
$$\text{,,} \quad 4\text{th}=120\times\frac{1}{12}=10.$$

The number 120 is said to be divided here *into parts proportional to the numbers* 6, 3, 2, 1, or *into parts in the ratio of* 6, 3, 2, 1. Also, the first part is said to be to the second as 6 is to 3, or in the ratio of 6 to 3.

Example 5. Divide £1180 into shares which shall be proportional to the reciprocals of 2, 5, 7.

The reciprocals of 2, 5, 7 are $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{7}$, or $\frac{35}{70}$, $\frac{14}{70}$, $\frac{10}{70}$; and the sum of these being $\frac{59}{70}$, we see that out of every $\frac{59}{70}$ of the whole $\frac{35}{70}$ must go to the first share, $\frac{14}{70}$ to the second, and $\frac{10}{70}$ to the third. Hence,

$$\text{1st share}=£1180\times\frac{\frac{35}{70}}{\frac{59}{70}}$$
$$=£1180\times\frac{35}{59}=£700$$
$$\text{2nd share}=£1180\times\frac{14}{59}=£280$$
$$\text{3rd share}=£1180\times\frac{10}{59}=£200.$$

Example 6. Divide 30 guineas among A, B, and C, so that A's share may be half as much as B's, and C's half as much again as A's and B's together.

Here for every £1 which B receives, A must receive £$\frac{1}{2}$, and C must receive £1 + £$\frac{1}{2}$ and half as much more, that is, in all £$\frac{3}{2}$ + £$\frac{3}{4}$, or £$2\frac{1}{4}$. The money has thus to

be divided in shares proportional to $\frac{1}{2}$, 1, $2\frac{1}{4}$; hence, since $\frac{1}{2}+1+2\frac{1}{4}=\frac{15}{4}$,

$$\text{A's share}=30\text{ guin.}\times\frac{\frac{1}{2}}{\frac{15}{4}}$$
$$=30\text{ guin.}\times\frac{2}{15}=4\text{ guin.}$$
$$\text{B's share}=30\text{ guin.}\times\frac{4}{15}=8\text{ guin.}$$
$$\text{C's share}=30\text{ guin.}\times\frac{9}{15}=18\text{ guin.}$$

Example 7. The sum of £100 10s. is to be divided among 12 men, 10 women, and 15 children, each woman receiving three-fourths of a man's share, and each child half of a woman's. Find what each man, each woman, and each child will receive.

$$1\text{ woman's share}=\frac{3}{4}\text{ man's share}$$
$$\therefore 10\text{ women's shares}=7\tfrac{1}{2}\text{ men's shares};$$
$$\text{and }1\text{ child's share}=\frac{1}{2}\text{ woman's share}$$
$$=\frac{3}{8}\text{ man's share}$$
$$\therefore 15\text{ children's shares}=5\tfrac{5}{8}\text{ men's shares.}$$

Hence $12+7\frac{1}{2}+5\frac{5}{8}$ men's shares = the total amount,

i.e., $25\frac{1}{8}$ men's shares = £100 10s.

$$\therefore 1\text{ man's share}=\pounds 100\ 10\text{s.}\div 25\tfrac{1}{8}$$
$$=\pounds 100\ 10\text{s.}\times\frac{8}{201}$$
$$=\pounds 4.$$

$$\text{Consequently, }1\text{ woman's share}=\pounds 4\times\frac{3}{4}=\pounds 1$$
$$\text{and }1\text{ child's }\text{,,}\ =\pounds 1\times\frac{1}{2}=10\text{s.}$$

Example 8. On starting business, A has a capital of £6000; B is admitted as a partner 4 months afterwards with a capital of £4000; and at the end of the year the gain amounts to £2900. What share of this should each receive, supposing that the rate of gain before the admission of B was a half more than it was afterwards?

A had £6000 invested for 4 mo. at the high rate,
and £6000 ,, 8 mo. at the low rate,

which is the same as if

A had £24000 invested for 1 mo. at the high rate,
and £48000 „ „ low rate,

which is the same as if

A had £24000 × $1\frac{1}{2}$ invested for 1 mo. at the low rate,
and £48000 „ „ „

which is the same as if

A had £84000 invested for 1 mo. at the low rate.

Again,

B had £4000 invested for 8 mo. at the low rate,

which is the same as if

B had £32000 invested for 1 month at the low rate.

The gain has thus to be divided into parts proportional to 84000 and 32000 so that

$$\text{A's share} = £2900 \times \frac{84000}{116000}$$
$$= £2100$$
$$\text{and B's share} = £2900 \times \frac{32000}{116000}$$
$$= £800.$$

Exercises. Set CVI.

1. A bankrupt's assets amount to £600, and his debts are £1200 to A, £200 to B, and £100 to C. What sum should each creditor receive?

2. Four partners contribute £600, £800, £1000, £1200 respectively to the capital of their joint undertaking. What fraction of the annual profits should come to each?

3. Three farmers agree to buy a road engine for £200, one of them to have the use of it for 6 months of the year, another for 4 months, and the third for 2 months. What share of the costs should each bear?

4. A fortune is bequeathed to three persons in parts proportional to 5, 3, 2, but the testator is found to have left a debt of £1650. How much less will each legatee thus receive?

5. Divide £39 18s. among three persons, so that as often as the first gets a crown, the second may get half-a-crown, and the third a florin.

6. Of the capital of an undertaking A subscribes £400, B £350, and C £250. What percentage of the gain should each receive?

7. Divide the sum of £413 2s. 8d. into two shares in the ratio of 7 to 9.

8. Divide 60 into parts proportional to the second powers of the first four integers.

9. One of the three partners of a trading firm owns a third of the capital, another owns a fourth, and the third owns the remainder. Find their shares of a year's profit amounting to £2117.

10. One kind of gunpowder is formed of 77 parts of nitre, 9 of sulphur, and 14 of charcoal. In $1\frac{1}{4}$ ton of it how much is there of each ingredient?

11. Divide 50 into parts proportional to the reciprocals of the first four integers.

12. Divide 41 into parts proportional to the second powers of the reciprocals of the first four integers.

13. Divide 41 into parts proportional to the reciprocals of the second powers of the first four integers.

14. The annual expense of keeping a horse is borne by four persons, who pay £20 3s. 4d., £19 10s., £15 16s. 8d., £5 6s. 8d. respectively. How many days of the year ought each person to have him?

15. Three persons obtain a lease of a field of 13 ac. 2 ro. 37 po. for gardening, and pay respectively £3 6s. 6d., £4 12s. 8d., £10 7s. of the rent. What share of the field should each claim?

16. A merchant becomes bankrupt and is owing money to four creditors, viz., to A £113 6s. 8d., to B £251 10s., to C £312 2s. 6d., and to D £505 10s. 10d. The bankrupt's assets being £739 1s. 3d., what share should each creditor receive?

17. One gentleman gives away 6 guineas, dividing his gift into three shares proportional to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$; another gives away $\frac{1}{2}$ of 6 guineas to one person, $\frac{1}{3}$ to another, and $\frac{1}{6}$ to another. Find the differences between the former shares and the latter.

18. Three graziers rent a field for 17 weeks at a cost of £29 8s. One of them keeps 20 oxen in it for 7 weeks, the second 25 oxen for 10 weeks, and the third 10 oxen the whole time. What share of the rent should each pay?

19. A fishing-rod $11\frac{1}{3}$ feet long is made up of four pieces, two of which are of the same length, the other two being respectively a half and a third of this length. How long is each piece?

20. A legacy of £452 13s. is divided among three sons, the share of the eldest being thrice that of the youngest and twice that of the second son. Find the shares.

21. A stationer starts business with a capital of £1000, and after 4 months takes a partner who supplies £2000. What share should each receive of the first year's profits, which amount to £1498?

22. Divide 210 apples among A, B, and C, so that B's share may be double of C's and half of A's.

23. Two partners, A and B, begin business together. Find A's share of the first year's gain of £1600, it having been agreed that as manager he is to receive a fixed salary of £500 and 10 per cent. more of the remainder than B.

24. Divide 56 oranges among 6 boys and 11 girls, giving each girl twice as many as each boy.

25. What share of £860 14s. 4d. should each of three persons receive, if the first is to have 25 per cent. more than the second, and the second 20 per cent. more than the third?

26. A tradesman commences business on 14th March with a capital of £500; on 1st June he takes a partner with £400; on 30th November another partner is admitted with £450; and at the end of the year a profit of £620 falls to be divided. Find the share of each.

27. The sum of £19 is divided among 24 men and 36 women, the shares of 3 men being equal to the shares of 5 women. Find each man's and each woman's share.

28. Divide £819 among A, B, C, D, E, so that A's share may be double of B's, treble of C's, half of D's, and half as much again as E's.

29. Three cattle-dealers rented a grass field for £30 9s. One of them had 10 oxen grazing in it for three weeks, another had 35 sheep for 4 weeks, and the third 5 oxen and 15 sheep for 2 weeks. Supposing that an ox eats three times as much as a sheep, what share ought each dealer to pay of the rent?

30. The sum of £23 3s. 10d. is paid to 14 men and 25 women, and for every 4d. given to each woman 6d. is given to each man. Calculate a man's share.

31. Three workmen receive in all three half-crowns, the share of the first being double that of the second, and half as much as those of the second and third together. Find the shares.

32. A piece of work is begun by three gangs of labourers, consisting of 12, 15, and 20 men respectively. The first gang remains 5 days at the work, the second remains 3 days longer, and the third finishes the work in 2 days more. What fraction of the whole is performed by one man in each gang?

33. Divide 220 apples among 12 men, 14 women, and 15 children, so that the shares of 3 men may be equal to the shares of 5 women, and the shares of 2 women equal to the shares of 3 children.

34. A sum of £30 is to be divided among 3 pedestrians according to their speed of walking, and they are found to travel a mile in 8 min. 30 sec., 8 min. 45 sec., and 9 min. respectively. Find the share of each.

35. The thickness of 10 cards is .1475 in., and 5 of them are of one thickness, 2 of them are each a half thicker than any of the five, and the remaining 3 are each half as thick again as either of these two. Find the thickness of one of each set.

36. Three partners, A, B, C, share a gain of £612 11s. B has had twice as much money as A in the business, but only for a third of the time, and C, who has been a partner for the same time as B, has had as much as A and B together. Find each partner's share of the gain.

37. A tradesman engages in business with a capital of £500, and

after 3 months takes a partner with a capital of £800. The gain for the first 6 months being £1200, how should it be divided, if the rate of gain per cent. per ann. during the partnership were double what it was before?

38. Two booksellers engage in trade as partners, the senior partner having a third more capital than the other. After 5 months both of them double their capital, and a third partner is admitted with a third of the capital held by the junior partner on starting. How should the year's gain of £707 be shared?

39. A church collection composed of threepenny, fourpenny, and sixpenny pieces amounted to £16 4s., and there was a third more threepenny pieces and a third less sixpenny pieces than fourpenny pieces. Find the number of each coin.

40. A begins business with a capital of £640; after 3 months B is admitted as a partner with £480; and after other 3 months C is admitted with £320. If the rate of gain per cent. is a third higher after the admission of B, and falls to its former position when C is admitted, what fraction of the year's gain should each receive?

160. Area of Surfaces.—We have already seen that a square whose side is 3 feet long contains 3 times 3 square feet; that one whose side is 12 inches long contains 12 times 12 square inches; and the learner may also already know that we speak of the 9 square feet and the 144 square inches as the *area* of the respective squares, the area of a surface being an expression for the quantity of it in square miles, square yards, or any other such units. The calculation of areas, of which we have here two examples, is dependent upon the science of geometry, and as the shapes and properties of surfaces are very varied, it is an exceedingly extensive subject. There is one form of plane surface, however, which is of very common occurrence in ordinary life, and the area of which is calculated in a way that can be explained without assuming much knowledge of geometry. This plane figure is the *rectangle*, of which the square may be considered a particular case.

In geometry, a rectangle is defined as a plane figure bounded by four straight lines, called its *sides*, and having all its angles right angles, *e.g.*, the figure A B C D; and it is shown that the opposite sides are equal. If a rectangle

has *all* its sides equal, it is called a square, *e.g.*, the figure P Q R S. The length of either of the two longer sides is called the *length* of the rectangle, the length of either of the two shorter its *breadth*, and both are spoken of as its *dimensions*.

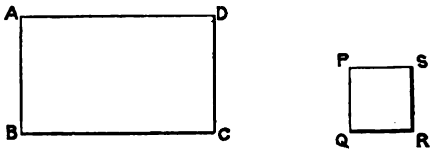

Fig. 6.

The area of a rectangle is usually calculated from knowing its length and breadth. Consider the case of a rectangle 8 in. long and 5 in. broad, and let the above figure A B C D represent it. Then, since A D is 8 in. long, it may be divided into 8 parts, each an inch in length, and we may draw lines through the points of division in the direction of the breadth. Similarly, A B may be divided into 5 such parts, and lines drawn lengthwise through the points of division, the result being as in the annexed diagram. Now each of the little figures thus formed must be a square inch, and there being 5 rows of them, and 8 in each row, it is clear that the number of square inches in the rectangle is 8×5; so that the area of a rectangle 8 in. long and 5 in. broad is (8×5) square inches.

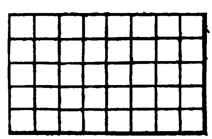

Fig. 7.

Had the dimensions been 8 *feet* and 5 *feet*, the area would evidently have been (8×5) *square feet*, and similarly in the case of any other unit. Further, what is established here when the numbers used in specifying the dimensions are 8 and 5, will be seen to hold in the case of any other whole numbers whatever. Thus the area of a rectangle, whose length is 7 miles, and breadth 2 miles, is 14 square miles;

and the area of a rectangle, whose length is 9 metres, and breadth 4 metres, is 36 square metres.

Coming now to the case where the dimensions are given in fractional numbers, let us take, as an example, a rectangle $4\frac{3}{4}$ yards long and $2\frac{1}{3}$ yards broad. Here the denominators of the two fractions being 4 and 3, we can express the dimensions in *twelfths* of a yard by means of whole numbers, the length being 57 twelfths of a yard, and the breadth 28. Now we already know that the area of a rectangle 57 twelfths of a yard long and 28 broad is (57×28) small squares, whose sides are a twelfth of a yard in length; and there being 144 of these in a square whose side is a yard in length, that is, in a square yard, the area in question is $\frac{57 \times 28}{144}$ sq. yd. But $\frac{57 \times 28}{144} = \frac{57}{12} \times \frac{28}{12} = 4\frac{3}{4} \times 2\frac{1}{3}$; so that the area of a rectangle whose length is $4\frac{3}{4}$ yd. and breadth $2\frac{1}{3}$ yd. is $(4\frac{3}{4} \times 2\frac{1}{3})$ sq. yd.

A similar result will be found in other instances, and we thus come to the general conclusion that whether the numbers used in expressing the length and breadth of a rectangle in terms of the same unit be integral or fractional, their product is the number of corresponding square units in the area.

The modes of calculating the areas of other figures are dependent upon this fundamental result and the theorems of geometry; but even without a knowledge of the latter, they are well worth remembering. Thus, the area of a *triangle* is shown to be half that of a rectangle with the same base and height; the area of a *circle* to be the area of the square on its radius multiplied by 3.14159...., or approximately by $\frac{22}{7}$, or $\frac{355}{113}$; and so on.

Example 1. The length of a rectangular black-board is 5 ft. 6 in., and the breadth 3 ft. $8\frac{1}{2}$ in. Find the area of its surface.

Length = 66 in.
and breadth = $44\frac{1}{2}$ in.

$$\therefore \text{area} = (66 \times 44\tfrac{1}{2}) \text{ sq. in.}$$
$$= 2937 \text{ sq. in.}$$
$$= 20 \text{ sq. ft. } 57 \text{ sq. in.}$$

Or thus:—

$$\text{Length} = 5\tfrac{1}{2} \text{ ft.}$$
$$\text{and breadth} = 3\tfrac{17}{24} \text{ ft.}$$
$$\therefore \text{area} = (5\tfrac{1}{2} \times 3\tfrac{17}{24}) \text{ sq. ft.}$$
$$= \frac{979}{48} \text{ sq. ft.}$$
$$= 20 \text{ sq. ft. } 57 \text{ sq. in.}$$

Example 2. Calculate the area, in acres, &c., of a rectangular field whose length is 25 chains 47 links, and breadth 16 chains 8 links.

$$\text{Length} = 25.47 \text{ chains}$$
$$\text{and breadth} = 16.08 \text{ chains.}$$
$$\therefore \text{area} = (25.47 \times 16.08) \text{ sq. chains}$$
$$= 409.5576 \text{ sq. chains}$$
$$= 40.95576 \text{ ac.}$$
$$= 40 \text{ ac. } 3 \text{ ro. } 33 \text{ po. nearly.}$$

Example 3. The area of a rectangular park is 173 ac. 2 ro. 23 sq. po. 13 sq. yd., and the breadth is 2 fur. 35 po. 3 yd. Find its length.

$$\text{Area} = 173 \text{ ac. } 2 \text{ ro. } 23 \text{ sq. po. } 13 \text{ sq. yd.}$$
$$= 840448\tfrac{3}{4} \text{ sq. yd.}$$
$$\text{and breadth} = 2 \text{ fur. } 35 \text{ po. } 3 \text{ yd.}$$
$$= 635\tfrac{1}{2} \text{ yd.}$$

Now we know that whatever may be the number of yards in the length, that number multiplied by 635½ must give 840448¾: hence

$$\text{length} = \frac{840448\frac{3}{4}}{635\frac{1}{2}} \text{ yd.}$$
$$= 1322\tfrac{1}{2} \text{ yd.}$$
$$= 6 \text{ fur. } 2\tfrac{1}{2} \text{ yd.}$$

Example 4. What length of gilt ribbon ¾ in. broad will be required to gild a circular plate 5½ in. in diameter?

$$\text{Radius of plate} = 5\tfrac{1}{2}\text{ in.} \div 2 = 2\tfrac{3}{4}\text{ in.}$$

$$\therefore \text{ area} \quad ,, \quad = \left(2\tfrac{3}{4} \times 2\tfrac{3}{4} \times \frac{355}{113}\right) \text{ sq. in. nearly,}$$
$$= \frac{42955}{1808} \text{ sq. in.}$$

Now, to cover the plate the piece of ribbon required must have the same area as the plate.

$$\therefore \text{ area of ribbon} = \frac{42955}{1808} \text{ sq. in.};$$

$$\text{but breadth} \quad ,, \quad = \frac{3}{4} \text{ in.}$$

$$\therefore \text{ length} \quad ,, \quad = \left(\frac{42955}{1808} \div \frac{3}{4}\right) \text{ in.}$$
$$= \frac{42955}{1356} \text{ in.}$$
$$= 2 \text{ ft. } 7\tfrac{919}{1356} \text{ in.}$$

Exercises. Set CVII.

1. Calculate the area of a rectangular ceiling 20 ft. 4 in. long and 13 ft. 7 in. broad.

2. A rectangular surface is 4 yd. 2 ft. 7 in. long and 1 yd. 1 ft. 1 in. broad. Find its area.

3. How many square feet of surface are there in a rectangular floor which is $16\frac{1}{3}$ ft. long and $12\frac{1}{3}$ ft. broad?

4. Find the area of a rectangular plate 4.105 in. broad and 1.625 ft. long.

5. A rectangular piece of ground is $113\frac{1}{2}$ ft. long and $17\frac{1}{4}$ ft. broad. Give its area in square yards.

6. Find the area in acres, &c., of a square field the length of whose side is 46 chains 25 links.

7. Express in acres, &c., the area of a rectangular field 1 fur. 13 po. $2\frac{1}{2}$ yd. long and 25 po. 4 yd. broad.

8. The surface of a rectangular table contains 27 sq. ft. 34 sq. in. and its length is 6 ft. 2 in. Find its breadth.

9. 400 rectangular tiles, each $2\frac{1}{2}$ in. long and $1\frac{1}{4}$ in. broad, are required to cover a hearth. Find the number of square feet in the hearth.

10. A rectangular piece of ground 30 po. in length and 6 po. $2\frac{3}{4}$ yd. in breadth is divided into 13 plots. Find how many square yards there will be in each plot.

11. How broad must a rectangular walk 116 yd. 2 ft. long be, when 105 flags $3\frac{1}{2}$ ft. long and $2\frac{6}{7}$ ft. broad are required to pave it?

12. What must be the breadth of a rectangular plantation whose area is $12\frac{1}{2}$ ac. and length 31 chains 25 links?

13. How many portions, each containing 72 sq. in., could be cut out of a piece of cardboard 24 in. square?

14. A circular plot of ground has a radius of 106 links. Find its area in acres, &c.

15. There is $1\frac{2}{3}$ sq. ft. of surface in the two faces of a rectangular school slate, and the breadth of the slate is 8 inches. Calculate its length.

16. From a book-leaf 12 in. long and $8\frac{1}{2}$ in. broad two equal rectangular pieces are cut, the dimensions of which are half those of the leaf. How many square inches of paper remain?

17. On a circular piece of ground 130 yd. in diameter a walk 3 yd. broad is made all round close to the boundary. Find the area of the walk.

18. Calculate the number of square feet of wall surface in a room whose circuit is 56 ft. 6 in. and height 11 ft. 8 in.

19. On the back of an envelope $4\frac{1}{8}$ in. long and $3\frac{1}{4}$ in. broad are gummed 4 postage stamps, each .9875 in. long and .8125 in. broad. What space remains for the address?

20. A drawing occupies one side of a sheet of paper 8 in. long and $6\frac{1}{4}$ in. broad, with the exception of a margin $\frac{3}{4}$ in. wide all round. How many square inches does the drawing fill?

21. How many yards of ribbon $3\frac{1}{3}$ in. wide would cover a square mile of surface?

22. Two circular flower-beds 15 ft. 6 in. in diameter are marked off in a rectangular plot of ground 60 ft. long and $52\frac{1}{2}$ ft. broad, and the remainder of the plot is paved. Calculate the area of the pavement.

23. What length of carpet $\frac{3}{4}$ yd. wide would be required to cover a rectangular floor 18 ft. long and $13\frac{1}{2}$ ft. broad?

24. A map is $6\frac{1}{4}$ in. in length and $4\frac{3}{8}$ in. in breadth, and the scale is $\frac{1}{8}$ in. to the mile. Find how many square miles of surface it represents.

25. What length of paper 22 in. wide would be required to cover a wall 16 ft. 6 in. long and 10 ft. 8 in. high?

26. A rectangular room is 16 ft. 6 in. long, 13 ft. 9 in. broad, and 11 ft. high. Find the total area of its wall surface.

27. Find the number of square yards of surface in the walls and ceiling of a rectangular chamber 20 ft. 4 in. long, 18 ft. 6 in. broad, and 10 ft. 2 in. high.

28. A circular flower-bed 12 ft. in diameter touches the sides of the square plot within which it is situated. Find the area of the remainder of the plot.

29. A rectangular garden 240 yd. long and 180 yd. broad is divided into two equal parts by a straight pathway 5 ft. broad, which runs in the direction of the length. Find the area of one of the parts.

30. The height of a rectangular room is 12 ft. 3 in., the length 27 ft. 4 in., and the breadth 18 ft. 6 in. Find the amount of wall space, allowing for three rectangular windows each 6 ft. 3 in. by 5 ft. 4 in.

31. From a rectangular piece of ground $16\frac{1}{2}$ yd. long and $12\frac{1}{3}$ yd. broad there is marked off a border path $3\frac{1}{2}$ ft. broad all round. What fraction is the path of the whole?

32. A map 3 in. long and 4 in. broad represents 48 square miles of the earth's surface. To what scale is it drawn?

33. How many squares $\frac{1}{2}$ in. on the side could be cut out of a rectangular piece of tin $3\frac{1}{2}$ ft. long and 10 in. broad?

34. A patchwork covering has an area of $4\frac{1}{6}$ sq. yd., and is composed of 150 square pieces of the same size. Find the length of a side of each of these squares.

35. A rectangular box $3\frac{1}{2}$ ft. long, 1 ft. $10\frac{1}{2}$ in. broad, and 1 ft. $\frac{1}{2}$ in. deep is to be covered on all sides with leather. How many square feet of leather will be required?

36. What would be the cost of polishing a rectangular block of granite 4 ft. long, $2\frac{1}{4}$ ft. broad, and $1\frac{1}{2}$ ft. thick, at the rate of 10d. per square foot?

37. How many slabs 4 ft. 3 in. long and 3 ft. broad would be required to pave a square surface whose side is 204 yd. long?

38. A sheet of paper 3 ft. 4 in. long and 1 ft. 5 in. wide is folded so as to form a pamphlet of 16 pages. Find the area of one page.

39. A rectangular surface $7\frac{1}{2}$ yd. long and $2\frac{1}{2}$ feet broad has its length diminished by $1\frac{1}{2}$ ft. By how much must the breadth be increased so as to preserve the same area as before?

40. An iron chest, the external dimensions of which are 4 ft. 2 in., 2 ft. 6 in., 1 ft. 8 in., and the walls of which are uniformly $\frac{3}{4}$ in. thick, has been painted both inside and outside. Calculate the total amount of painted surface.

41. The length of a rectangular room is 24 ft. 7 in., the breadth 20 ft. 5 in., and the area of the walls and ceiling 1851 sq. ft. 131 sq. in. Find the height of the room.

42. The length of a rectangular room is 16 ft. 8 in., the height 14 ft., and the area of the walls and ceiling $130\frac{1}{81}$ sq. yd. Find the breadth of the room.

43. The total superficial area of a rectangular block of gold is 15.112 sq. in., and two of its dimensions are 1.4 in. and 1.04 in. Find the third.

161. BULK OF SOLIDS.—The *bulk* of a solid body is the quantity of space it occupies expressed in cubic inches, cubic feet, or any other such units. Solids, like surfaces, are infinitely various in form; and, as before, we consider the calculation of the bulk of those only which are the simplest and most common, viz., the correlatives in solid geometry of the rectangle and square.

A *rectangular solid*, or *rectangular parallelopiped*, is a

figure bounded by six rectangular surfaces, called its sides or faces; for example, a brick, or a box, in their ordinary forms. The opposite sides of the figure are known to be alike. If *all* be alike, that is, if they be all squares, the figure is called a *cube*. Rectangular solids are seen to have length, breadth, and thickness, that is to say, *three* dimensions.

The calculation of the bulk of a rectangular solid is usually made from knowing the length, breadth, and thickness. Consider the case represented in the accompanying woodcut, where the length, B C, is 8 in., the breadth, C D, 5 in., and the thickness, A B, 3 in. Dividing A B into three

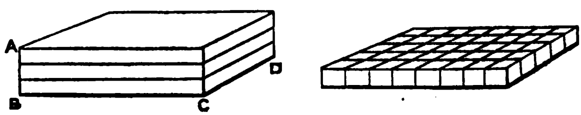

Fig. 8.

equal portions, and cutting, as indicated, through the points of division, parallel to the upper face, we get 3 slices, each 1 in. thick, 8 in. in length, and 5 in. in breadth. Now, taking one of these, we see that by making 8 cuttings in one direction, and 5 in another, as shown in the second woodcut, we divide it into 40 cubic inches, so that in the whole 3 slices the number of cubic inches is 40×3, or $8 \times 5 \times 3$. The bulk, therefore, of a rectangular solid, whose dimensions are 8 in., 5 in., 3 in., is seen to be $(8 \times 5 \times 3)$ cubic inches.

In the same way we may show that the bulk of a rectangular solid 4 yd. long, 3 yd. broad, and 2 yd. thick, is $(4 \times 3 \times 2)$ cubic yards; and the learner will perceive that a similar result must be obtained in every case where the unit of length, whatever it may be, is contained an exact number of times in each of the dimensions. Further, by treating, as on p. 276, the case where fractional numbers occur, he may convince himself, as before, of the truth

of the general conclusion, viz., that whether the numbers used in expressing the dimensions of a rectangular solid in terms of the same unit be integral or fractional, their product is the number of corresponding cubic units in the bulk.

Of the many results deduced from this by means of the theorems of solid geometry, the learner may remember one, viz., that the bulk of a *sphere* or *ball* is the same as the bulk of the cube on the radius multiplied by $\frac{4}{3}$ of 3.14159.... Also it may be noted that *volume* and *cubic content* are sometimes used for bulk, *width* in place of breadth, and *depth* and *height* instead of thickness.

Example 1. On a flat piece of ground, containing 15 sq. ft. 36 sq. in., a rectangular block of brickwork, 10 ft. 3 in. high, is built. Find its bulk.

Area of surface $=2196$ sq. in.
and height $=123$ in.

$\therefore$ bulk $=(2196\times 123)$ cub. in.
$=270108$ cub. in.
$=156$ cub. ft. 540 cub. in.

Or thus :—

Area of surface $=15\frac{1}{4}$ sq. ft.
and height $=10\frac{1}{4}$ ft.

$\therefore$ bulk $=(15\frac{1}{4}\times 10\frac{1}{4})$ cub. ft.
$=\frac{2501}{16}$ cub. ft.
$=156$ cub. ft. 540 cub. in.

Example 2. Find the bulk of a rectangular block of ice, 6 ft. 3 in. long, 2 ft. $4\frac{1}{2}$ in. broad, and $8\frac{5}{8}$ in. thick.

Length $=6\frac{1}{4}$ ft., breadth $=2\frac{3}{8}$ ft., thickness $=\frac{23}{32}$ ft.

$$\therefore \text{bulk}=\left(6\frac{1}{4}\times 2\frac{3}{8}\times \frac{23}{32}\right) \text{ cub. ft.}$$
$$=\left(\frac{25}{4}\times\frac{19}{8}\times\frac{23}{32}\right) \text{ cub. ft.}$$
$$=\frac{10925}{1024} \text{ cub. ft.}$$
$$=10 \text{ cub. ft. } 1155\tfrac{15}{16} \text{ cub. in.}$$

Or thus :—

$$\begin{aligned} \text{Bulk} &= (75 \times 28\tfrac{1}{2} \times 8\tfrac{5}{8}) \text{ cub. in.} \\ &= 18435\tfrac{15}{16} \text{ cub. in.} \\ &= 10 \text{ cub. ft. } 1155\tfrac{15}{16} \text{ cub. in.} \end{aligned}$$

Example 3. A tank $17\frac{1}{2}$ yd. long and $4\frac{4}{7}$ yd. broad, is to be capable of containing 540 cub. ft. of water. What must its depth be?

$$\begin{aligned} \text{Volume of tank} &= 540 \text{ cub. ft.} \\ &= 20 \text{ cub. yd.} \\ \text{and area of bottom} &= (17\tfrac{1}{2} \times 4\tfrac{4}{7}) \text{ sq. yd.} \\ &= \left(\frac{35}{2} \times \frac{32}{7}\right) \quad " \\ &= 80 \quad " \end{aligned}$$

Now we know that the number of yards in the depth must be such a number that when we multiply 80 by it, we shall have 20 for result; hence

$$\begin{aligned} \text{depth} &= (20 \div 80) \text{ yd.} \\ &= \frac{1}{4} \text{ yd.} \end{aligned}$$

Example 4. $\frac{1}{8}$ cub. ft. of wax is run into a cake $\frac{1}{4}$ in. thick. Find the extent of the surface of this cake.

$$\begin{aligned} \text{Bulk of wax} &= \frac{1}{8} \text{ cub. ft.} \\ &= 216 \text{ cub. in.} \\ \text{and thickness of cake} &= \frac{1}{4} \text{ in.} \end{aligned}$$

and whatever may be the number of square inches in the surface, we know that the number multiplied by $\frac{1}{4}$ must give 216.

$$\begin{aligned} \therefore \text{ area of surface} &= (216 \div \tfrac{1}{4}) \text{ sq. in.} \\ &= 864 \text{ sq. in.} \\ &= 6 \text{ sq. ft.} \end{aligned}$$

EXERCISES. SET CVIII.

1. Calculate the cubic content of a rectangular box whose internal dimensions are 2 ft. $1\frac{1}{2}$ in., $10\frac{1}{2}$ in., and 1 ft.

2. How many cubic feet of stone are there in a rectangular slab whose length is $2\frac{1}{2}$ yd., breadth $2\frac{1}{2}$ ft., and thickness $2\frac{1}{2}$ in.?

3. The cross section of a rectangular beam of wood contains 2 sq. ft. 110 sq. in., and the length of the beam is $2\frac{1}{2}$ yd. Find its bulk.

4. A rectangular hollow is to be dug in a level piece of ground, the dimensions being 100 ft., $28\frac{1}{2}$ ft., and $1\frac{1}{2}$ ft. Calculate how many cubic yards of earth must be removed.

5. The length of a rectangular block of building stone is $1\frac{1}{2}$ yd., the breadth 1 ft. 6 in., and the thickness $10\frac{1}{2}$ in. Find its bulk.

6. How many cubes with an edge of 3 in. could be cut out of a cube with an edge of 3 ft.?

7. The dimensions of a rectangular plate of silver are 1.25 in., .125 in., .0125 in. Find its bulk.

8. The length, breadth, and bulk of a rectangular mass of concrete are $3\frac{1}{2}$ ft., $2\frac{2}{3}$ ft., 36288 cub. in. respectively. Find the thickness.

9. 2000 bricks, each 9 in. long, $4\frac{1}{2}$ in. broad, and 3 in. thick, are required to fill a waggon. Find the cubic content of the waggon.

10. The length of a rectangular school-room is 50 ft. 6 in., the breadth 16 ft., and the height 10 ft. How many pupils could it accommodate, allowing according to the government regulations 80 cub. ft. of space for each pupil?

11. How many cubic inches are there in a rectangular solid whose length is $1\frac{1}{2}$ yd. and whose cross section is a square 1.45 ft. on the side?

12. There are four equal cubes whose edge is $\frac{1}{8}$ of an inch in length, and one cube whose edge is twice as long. Find the difference in bulk between the former and the latter.

13. The cross section of a rectangular plank of wood measures $4\frac{1}{2}$ in. by $2\frac{1}{4}$ in. In order to have a cubic foot of timber what length must I cut off?

14. How many portions, each containing $1\frac{1}{2}$ cub. in., are contained in a cube whose edge is half a yard long?

15. If the earth were actually instead of approximately a solid sphere 3956 miles in radius, how many cubic miles of matter would it contain?

16. How many cubes .01 in. on the side would be of the same bulk as a cube .1 ft. on the side?

17. A rectangular gas-holder is internally 12 ft. long, $5\frac{1}{2}$ ft. broad, and $4\frac{1}{2}$ ft. deep. How many more cubic feet of gas would it hold if each dimension were increased by half a foot?

18. The diameter of a spherical pellet of shot is an eighth of an inch. Find its bulk.

19. How many cubes whose edge measures $1\frac{1}{2}$ in. could be packed into a rectangular box whose dimensions are 1 ft. 3 in., $10\frac{1}{2}$ in., $4\frac{1}{2}$ in.

20. A brick wall the thickness of which is 14 in., and the face of which has $272\frac{1}{4}$ sq. ft. of surface, contains a *rod of brickwork*. How many rods are there in a rectangular wall whose dimensions are 180 ft., $6\frac{1}{2}$ ft., 1 ft. 11 in.?

21. A rectangular swimming-tank is a fathom deep and has a surface of half an acre. How many cubic feet of water does it contain?

22. A lead ball 6 in. in diameter is melted and recast into bullets ¼ in. in diameter. How many of the latter should there be?

23. A rectangular tin box is internally 8½ in. broad and 2¼ in. deep. What must its length be in order that it may hold a gallon?

24. A rectangular corn-chest measures internally 4 ft. 2 in. in length, 2 ft. 8 in. in breadth, and 2 ft. 6 in. in depth. How many pecks will it hold?

25. The leaves of a book are 7.125 in. long and 4.815 in. broad, and the thickness of 400 of them is three-quarters of an inch. Find the bulk of one of them.

26. Into a basin full of water three cubes of iron are thrown, the edges of which measure 4 in., 3½ in., 3¼ in. respectively. How many gills of water will overflow?

27. How many gallons of water must be run off from a rectangular cistern which is 4 ft. 3 in. long and 2 ft. 6 in. broad, to lower the surface of the water in it 10½ in.?

28. A spherical iron shell is 10 in. in diameter, the thickness of the iron being uniformly 2½ in. What room is there inside for explosives?

29. A rectangular tank receives 1400 gallons of oil, and the liquid surface level is thereby raised half a foot. Find the area of the bottom of the tank.

30. A cubic inch of steel is raised in temperature from 0° C. to 100° C., each dimension being thereby increased .001 in. How much is the bulk increased?

31. Into a cubical box full of water a ball 1 ft. 8 in. in diameter is put, and is found exactly to touch all the sides of the box. What bulk of water may be remaining?

32. A rectangular solid 4½ ft. long, 3½ ft. broad, 1⅓ ft. thick is increased 11 in. in thickness. By how much must the breadth be diminished so as to retain the same bulk as before?

33. The external dimensions of a rectangular iron chest are 2 ft. 3 in., 1 ft. 8 in., 1 ft. 2½ in., and the sides, lid, and bottom are 1 in. thick. Of how many cubic inches of iron is it formed?

34. The internal dimensions of a rectangular stone coffin without a lid are 6 ft., 1 ft. 8 in., 1 ft. 6 in., and the walls are uniformly 6½ in. thick. Of what bulk of stone is it composed?

35. 1000 gallons of water are run into an empty cylindrical tank, the area of the bottom of which is 2 sq. yd. 8 sq. ft. 84 sq. in. How high must the water stand in the tank?

36. A ball of lead is cast into a rectangular block equal in length and breadth to the former diameter, viz. 4 inches. Find the thickness of the block.

37. A cubical vessel is filled by a cubic foot of water, and from it enough is poured to fill another cubical vessel whose dimensions are a third less. How much is the level in the former vessel thereby lowered?

38. The bore of a pipe is 8½ sq. in. and water is running through it

at the rate of 5 ft. per second. How many cubic feet does it supply per day?

39. When the temperature of a cube of zinc is raised from 32° F. to 212° F., each dimension is thereby increased .3 per cent. Find the percentage of increase in the bulk.

40. A round tower is surmounted by a hemispherical roof. Calculate how many cubic feet of space there are in it, knowing that the internal diameter is 20 ft. and the height of the walls 35 ft.

THE ROOTS OF NUMBERS.

162. A POWER of a number, as has already been seen, is a product each of whose factors is the given number, and it is called the *second* power if the repeated factor occurs twice, the *third* power if the repeated factor occurs thrice, and so on. Thus

$$25, \textit{ i.e. } 5 \times 5, \text{ or } 5^2,$$

is called the second power of 5, and

$$\frac{27}{64}, \textit{ i.e. } \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}, \text{ or } \left(\frac{3}{4}\right)^3$$

is called the third power of $\frac{3}{4}$.

163. When one number is a power of another number the second is known as the corresponding ROOT of the first. Thus 64 being the second power of 8, 8 is called the second root of 64, and .001 being the third power of .1, .1 is called the third root of .001. A root of a number may thus be defined as a number the corresponding power of which is equal to the given number.

The root sign is $\sqrt{}$, a corruption of the letter *r*, and the particular root intended is indicated by a small figure placed over the first portion of the symbol: thus $\sqrt[5]{}$ $\sqrt[8]{}$ stand for *the fifth root of*, *the eighth root of*, respectively. When it is the second root that is meant to be specified the small figure or *index* is often omitted. Thus

$$\sqrt[3]{}.001 \text{ or } \sqrt[3]{(.001)} = .1,$$
$$\text{and } \sqrt{64} \text{ or } \sqrt[2]{64} = 8.$$

164. The finding of a particular power of a given number, or, as the expression is, the *raising* of a given number to a particular power, is an exercise in multiplication; the finding or *extraction* of a particular root of a given number is an inverse process requiring special attention. The two operations are sometimes spoken of as *involution* and *evolution*.

The learner may advantageously compare the problem of the *extraction* of a root of a number with the two other inverse problems of multiplication already dealt with, viz. *division* of one number by another, and the *resolution* or *decomposition* of a number into prime factors.

Extraction of the Second Root.

165. The problem of the extraction of the second root of a number being the finding of a number which when multiplied by itself produces the given number, it is soon seen that there are many numbers whose second roots very readily present themselves. Thus, knowing that the second powers of the first nine integers are

$$1, 4, 9, 16, 25, 36, 49, 64, 81,$$

we can at once tell the second root of each of this series of numbers; and if we happen to know any other such fact, as, *e.g.*, that

$$17 \times 17 = 289,$$

we know also that

$$\sqrt{289} = 17.$$

Further, observing that if we take a number and raise it to the second power, then take the number which is ten times as large and perform the same operation, the result in the latter case is *one hundred* times that in the former—thus

$$7 \times 7 = 49, \quad 70 \times 70 = 4900, \quad 700 \times 700 = 490000, \ldots\ldots$$

—it is clear that we can tell the second root of any number whose digits are the digit or digits of one of the second powers

$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,$$

followed by an even number of zeros.

Similarly, observing that if we take a number and raise it to the second power, then take a number which is a tenth of this and perform the same operation, the result in the latter case is *one-hundredth* of that in the former—thus

$$7\times7=49,\quad .7\times.7=.49,\quad .07\times.07=.0049,\ldots\ldots$$

—we see we can tell the second root of any number got from one of the second powers

$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,$$

by shifting the units' mark an even number of places to the left.

Lastly, since the second power of a fraction is got by finding the second power of the numerator and the second power of the denominator and forming the fraction whose numerator and denominator are respectively these second powers, it is evident that we can tell the second root of any fraction whose numerator and denominator are such that we happen to know their second roots. Thus, knowing that

$$\sqrt{16}=4 \text{ and } \sqrt{49}=7,$$

we also know that

$$\sqrt{\frac{16}{49}}=\frac{4}{7},$$

and that

$$\sqrt{3\tfrac{1}{16}},\ i.e.\ \sqrt{\frac{49}{16}}=\frac{7}{4},\ i.e.\ 1\tfrac{3}{4}.$$

EXERCISES. SET CIX.

What is the second root of

1. 400, 3600, 6400, 10000.
2. 90000, 1000000, 250000.
3. 1600000000, 81000000.
4. .25, .0025, .01, .0016.
5. .000081, .0009, .000001.
6. .00000036, .00000004.
7. $\frac{1}{4}, \frac{1}{25}, \frac{4}{81}, \frac{9}{64}$.
8. $\frac{16}{25}, \frac{25}{64}, \frac{16}{81}, \frac{64}{81}$.
9. $\frac{49}{100}, \frac{9}{1600}, \frac{1}{40000}, \frac{81}{4900}$.
10. $2\frac{1}{4}, 6\frac{1}{4}, 1\frac{7}{9}, 2\frac{7}{9}$.
11. $1\frac{9}{16}, 5\frac{1}{16}, 3\frac{6}{25}, 1\frac{32}{49}$.
12. $11\frac{1}{9}, 44\frac{4}{9}, 32\frac{32}{49}, 30\frac{70}{81}$.

Find by trial the second root of

13. 324, 625, 361. 14. 484, 961, 10201.

15. Between what two powers of 10 does the second root of 84681 lie? and between what two multiples of 10?

166. As the learner will have already observed, there are many integral numbers which are not the second powers of integers, *e.g.* the numbers 2, 3, 5,..... In proceeding to give a general process for the extraction of the second root, we leave such numbers out of consideration for the present, and deal first of all with numbers which, like 16, 81, 900,... are second powers of whole numbers.

167. When the second root of a given number is sought, the course which first suggests itself is that of Ex. 13, 14, Set CIX., viz. experiment. For example, the number being 1444, 32 might occur as likely to be the second root; we should therefore raise 32 to the second power, obtaining the result 1024, which shows that 32 is too small; then we should take a larger number than 32, raise it to the second power, and so on.

168. In order to improve upon this course of procedure, it is clearly important to have an answer to the following question, viz.: "If the number tried be so much more than 32, how much more than 1024 (*i.e.* 32^2) will its second power be?" for then we shall have a better chance of being successful in the second trial.

The answer to this general question will be suggested by considering a special case. Suppose the number tried be 36, *i.e.* $32+4$. We have then to find

$$(32 + 4)^2,$$

and must therefore multiply $32+4$ by itself. Now we are familiar with the fact that to multiply $32+4$ by $32+4$ we may multiply $32+4$ by 32 and $32+4$ by 4, and add the results. But the result of multiplying $32+4$ by 32 is got by taking the sum of 32×32 and 4×32, and therefore will be

$$32^2 + 4 \times 32.$$

Similarly the other result, viz. that got by multiplying $32+4$ by 4, is obtained by taking the sum of 32×4 and 4×4, and therefore will be

$$32 \times 4 + 4^2.$$

Hence the two results taken together amount to the sum of 32^2, 4×32, 32×4, and 4^2, and the second and third of these being equal, the sum may be put in the form

$$32^2 + 2 \times 32 \times 4 + 4^2.$$

We thus learn that the second power of $32+4$ is greater than the second power of 32 by twice 32 multiplied by 4 together with the second power of 4, and the question proposed is answered for the special case considered.

The second power of 4 and twice 32 multiplied by 4 amounting to 272, this gives $1024+272$, *i.e.* 1296, as the second power of 36, so that 36 like 32 is too small.

Taking now $32+6$ as likely to be correct, let us find its second power exactly in the manner preceding, but more shortly, thus:—

$$\begin{aligned}(32+6)^2 &= (32+6) \times (32+6) \\ &= (32+6) \times 32 + (32+6) \times 6 \\ &= 32^2 + 6 \times 32 + 32 \times 6 + 6^2 \\ &= 32^2 + 2 \times 32 \times 6 + 6^2.\end{aligned}$$

This, it will be found, is equal to 1444, so that the second root of the given number is 38; but the important point to be noticed is the general result which is evident from the two cases considered, and which gives the answer to the question proposed, viz.: *If the number tried be greater than* 32 *by an additional number* (4 or 6 in the above cases), *the second power will be greater than* 32^2 *by twice* 32 *times the additional number, together with the second power of the said additional number.*

169. A little examination will show that what is here stated in regard to 32 is equally true of any other number, so that, with a slight alteration in the mode of statement,

we have the following still more general and important truth, viz.: *The second power of the sum of two numbers* ($32+4$ and $32+6$ above) *is greater than the second power of one of the numbers by twice this number multiplied by the other, together with the second power of that other.* Thus, taking two additional cases where the sums of the two numbers are the same, viz. 17, we have

$$(12+5)^2 = 12^2 + 2 \times 12 \times 5 + 5^2 = 144 + 120 + 25$$
$$= 289,$$

$$\text{and } (10+7)^2 = 10^2 + 2 \times 10 \times 7 + 7^2 = 100 + 140 + 49$$
$$= 289.$$

170. The full value of this theorem lies not in its enabling us to find the second power of a number, but in helping us to find a better approximation to the second root when one approximation has been already tried. Suppose that 9025 is the given number, and that we have tried 85 as a number likely to be its second root, and have found it to be too small, 85^2 being $=7225$, which is 1800 less than the given number. The correct second root will therefore be so much more than 85 that its second power will be greater than 85^2 by 1800. Now we know otherwise that, since it exceeds 85, it must have its second power greater than 85^2 by twice 85 times its excess over 85, together with the second power of this excess. Consequently, twice 85 times the excess, together with the second power of the excess, must be equal to 1800, so that 170 times the excess must be nearly 1800, and therefore the excess itself nearly 10. Trying 10, we find that twice 85 times 10 and 10^2 amount exactly to 1800, so that the second root sought is $85+10$, *i.e.* 95.

If we had taken 90 as our first approximation to the second root, the work, on account of the decimal nature of our notation, would have been easier. Thus it is at once seen that $90^2=8100$, and that this is less than the given number by 925. Then as before we say that twice 90 times

the defect of 90 from the true result, together with the second power of this defect, must equal 925, so that 180 times the defect must nearly equal 925, and that therefore the defect itself must be nearly 5. Then trying, we find it to be exactly 5, and consequently the result is 90+5, *i.e.* 95, as before.

The figuring in these processes is sometimes arranged as follows :—

$$\begin{array}{rr|l} & 9025 & \underline{85+10} \\ 85^2= & \underline{7225} & \\ 2\times 85\times 10=1700 & 1800 & \\ 10^2=\underline{\ \ 100} & & \\ & \underline{1800} & \end{array} \qquad \begin{array}{rr|l} & 9025 & \underline{90+5} \\ 90^2= & \underline{8100} & \\ 2\times 90\times 5=900 & 925 & \\ 5^2=\underline{\ \ 25} & & \\ & \underline{925} & \end{array}$$

The finding of the 10 in the one case and of the 5 in the other is not shown, being done mentally on asking, in the former case how often 2×85 is contained in 1800, and in the latter how often 2×90 is contained in 925.

As another example, let us find the second root of 1369.

$$\begin{array}{rr|l} & 1369 & \underline{30+7} \\ 30^2= & \underline{900} & \\ 2\times 30\times 7=420 & 469 & \\ 7^2=\underline{\ \ 49} & & \\ & \underline{469} & \end{array}$$

Knowing that 30^2 is less than 1369, and 40^2 is greater, we see that the second root sought must lie between 30 and 40. Instead of taking any of the numbers 31, 32, ... we prefer from the experience before gained to begin with 30. Now 30^2 is less than 1369 by 469, and 469 being known to be 60 (*i.e.* 2×30) times the additional number wanted, together with the second power of this number, it follows that as the additional number is small as compared with 30, 60 times this number will be approximately equal to 469. Dividing 469 by 60 we get 7, and taking $2\times 30\times 7$ and 7^2 we find the amount to be exactly 469. Hence,

$$\sqrt{1369}=37.$$

EXERCISES. SET CX.

Find the second root of the following numbers :—

1. 529.	2. 1156.	3. 1936.	4. 2116.
5. 2809.	6. 4225.	7. 8281.	8. 5041.
9. 3481.	10. 2401.	11. 7744.	12. 9409.
13. $\frac{256}{1849}$.	14. $\frac{676}{1521}$.	15. $\frac{841}{1225}$.	16. $\frac{2209}{9801}$.
17. $1\frac{808}{961}$.	18. $1\frac{1755}{2601}$.	19. $3\frac{913}{2704}$.	20. $68\frac{89}{100}$.
21. 16.81.	22. 7.84.	23. .006241.	24. .00008464.

171. In the preceding exercises there are three or four digits in the given number, and the second root is attained in two steps: the number of tens in the root is got as a first step, and then as a second the perfect result. When the given number is expressed by more than four digits, a greater number of successive approximations is usually necessary, and these we are careful to choose in the way our system of notation renders desirable, viz. first the number of units of the highest kind in the result, then the number of units of the next highest kind, and so on. Thus, if the result were 4376, it would be obtained in four successive portions, 4000, 300, 70, 6, and not in such portions as 4138, 217, 19, 2, although it is possible so to proceed. It is thus important to be able easily to tell (1) what is the highest kind of unit in the result, and (2) how many there are of it; in other words, (1) how many digits the result contains, and (2) what is the first digit.

172. Knowing that

$$10^2 = 100$$
$$100^2 = 10000$$
$$1000^2 = 1000000$$
$$10000^2 = 100000000$$
$$\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots,$$

and observing that the numbers raised to the second power are the smallest numbers of 2, 3, 4, 5, ... digits, that the numbers on the right are the smallest numbers of 3, 5, 7, 9, . . . digits, and that the number of digits in the latter

case is always one less than twice the number of the digits in the former, we readily conclude that corresponding to 3 or 4 digits in the given number there are 2 digits in the second root, corresponding to 5 or 6 there are 3, corresponding to 7 or 8 there are 4, corresponding to 9 or 10 there are 5, and so on. The same will also be apparent on considering the ordinary process of multiplying a number by itself. For example, if the number contain 7 digits, when we multiply by the last of them we begin by passing over 6 places to the right of the units' mark, and then continue through either 7 or 8 places more, consequently there must be either 13 or 14 places in the result.

When the number of digits in the second root of a given number is known, it is not difficult to tell what the first digit is. For example, take the number 6985449. Here there are 7 digits, consequently there must be 4 digits in the second root sought, a fact which is usually made evident by marking off the original digits in pairs, beginning at the place of the units' mark, thus:—

6′98′54′49.

Now the highest unit in the root being the *thousand*, the question is how many thousands there are; and the answer evidently is 2 thousand, for the second power of this is 4′00′00′00, which is less than the given number, and the second power of 3 thousand is 9′00′00′00 which is greater. Again, if the number were 488601 we should separate the figures thus—

48′86′01,

thereby showing that the highest unit in the root is the *hundred;* and seeing that $600^2=36'00'00$ and $700^2=49'00'00$, we should conclude that the number of hundreds in the root is 6.

173. Proceeding now to the extraction of the second root in the case of numbers such as have just been referred to, let us consider as an example the number 760384.

The figuring arranged in the manner shown in § 170 is as follows :—

$$\begin{array}{rr|l}
 & 76'03'84 & \underline{800+70+2} \\
800^2= & \underline{64\,00\,00} & \\
2\times 800\times 70=112000 & 12\,03\,84 & \\
70^2=\underline{\quad 4900} & & \\
 & 11\,69\,00 & \\
\hline
2\times 870\times\ 2=\quad 3480 & 34\,84 & \\
2^2=\underline{\qquad 4} & & \\
 & \underline{34\,84} &
\end{array}$$

Here, beginning on the right hand, we mark off the digits in pairs, and find the first approximation to be 800. After subtracting 800^2 we inquire how often 1600 (*i.e.* 2×800) is contained in the remainder. Finding that it will be over 70, we take 70 as the next portion of the root, and having found the sum of twice 800 times 70 and 70^2 to be 116900, we subtract this from the remainder in question, which gives us the new remainder 3484. Now, having taken away 800^2 and $2\times 800\times 70+70^2$, we have subtracted in all 870^2 and thus know that the given number is greater than 870^2 by 3484. The stage we have reached is therefore exactly the stage we should have been at if we had at once tried 870 as our first approximation, the figuring in which case would have stood thus :—

$$\begin{array}{r|l}
760384 & \underline{870} \\
870^2=\underline{756900} & \\
3484 &
\end{array}$$

Moreover, if we had done this we should next have inquired how often 2×870 is contained in 3484, and have concluded the operation as follows :—

$$\begin{array}{r|l}
2\times 870\times 2=3480 & 3484 \\
2^2=\underline{\qquad 4} & \\
 & \underline{3484}
\end{array}$$

and this is exactly what we do in the other case also, as is

seen above. Thus in the process of extraction the third portion of the root is found exactly as the second is found; and so also are all the succeeding portions if there be any.

174. For practical purposes, however, the figuring employed in this process of extraction may be considerably curtailed. First of all, we may adopt the abridgment made in the ordinary processes of multiplication and division, viz. the omission of final zeros and omissions associated with this. The saving thereby obtained will be seen by comparing the following:—

(I.)

$$
\begin{array}{rr|l}
 & 76'03'84 & \underline{800+70+2} \\
800^2 = & \underline{64\,00\,00} & \\
2\times800\times70 = 112000 & 12\,03\,84 & \\
70^2 = \underline{\quad 4900} & & \\
 & 11\,69\,00 & \\
\hline
2\times870\times\ 2 = \quad 3480 & 34\,84 & \\
2^2 = \underline{\qquad 4} & & \\
 & \underline{34\,84} &
\end{array}
$$

(II.)

$$
\begin{array}{rr|l}
 & 76'03'84 & \underline{872} \\
8^2 = & \underline{64} & \\
2\times8\times7 = 112 & 12\,03 & \\
7^2 = \underline{\ 49} & & \\
 & 11\,69 & \\
\hline
2\times87\times2 = 348 & 3484 & \\
2^2 = \underline{\quad 4} & & \\
 & \underline{3484} &
\end{array}
$$

The second abbreviation is of a different kind, viz. the substitution of one set of operations for another set equivalent to the former but less desirable. Instead of multiplying 160 (*i.e.* 2×80) by 7, and 7 by 7, and adding the two results, as we do above, we may simply multiply 167 by 7; instead of multiplying 1740 (*i.e.* 2×870) by 2, and 2 by 2, and adding the results, we may simply multiply 1742 by 2;

and similarly with every such set of operations. Thus we have the form—

(III.)

$$\begin{array}{rl}
 & 76'03'84 \mid \underline{872} \\
 & \underline{64} \\
 & 1203 \\
167 \times 7 = & \underline{1169} \\
 & 3484 \\
1742 \times 2 = & \underline{3484}
\end{array}$$

and from this we may further omit the portions " $\times 7 =$," " $\times 2 =$."

175. The curtailment of the figuring is accompanied by a simplification of the mental work and a change in its character. Without keeping in view the reasons for the steps taken, we need now only proceed as follows:—

$$\begin{array}{r|l}
 & 6'98'54'49 \mid \underline{2643} \\
 & \underline{4} \\
46 & 298 \\
 & \underline{276} \\
524 & 2254 \\
 & \underline{2096} \\
5283 & 15849 \\
 & \underline{15849}
\end{array}$$

Having marked off the digits in pairs, we find on the left of the last mark the figure 6, and we ask ourselves what is the largest integer whose second power is less than 6. Taking note of this integer as the first digit of the root, we place its second power under the 6, and subtract. Then affixing to the remainder the two figures 98 which follow the 6 in the original number, and placing to the left of the remainder the double, 4, of the first figure of the root, we inquire how often 4 is contained in 29. The answer to this gives an approximation to the second digit of the root, which is not 7, as we should soon find, but 6; and we note

6 down immediately after the 2 of the root, and after the 4 which was placed on the left of 298. Then the 46 on the left is multiplied by the 6 of the root, and the result placed under the 298 and subtracted from it. To the remainder there is appended, as before, the next pair of figures in the original number, viz. 54, and to the left of the remainder there is put the double of the 26 which now stands on the right of the original number. Then we ask how often 52 is contained in 225, and so on, exactly as in the case of the preceding digit of the root.

176. When one or more zeros, followed by a significant figure, occur in the root, a little additional care is necessary. Having found that a particular digit must be 0, we put it down in the usual two places, but do not of course go on to multiply: then the next pair of figures in the original number is appended to the last pair written, and we are again ready to ask "How often?" and proceed as before. Thus, in finding $\sqrt{497025}$ and $\sqrt{25090081}$, the figuring would be as follows:—

$$\begin{array}{rl} & 49'70'25 \,\big|\, \underline{705} \\ & \underline{49} \\ 1405 & \big|\; 7025 \\ & \big|\; \underline{7025} \end{array} \qquad\qquad \begin{array}{rl} & 25'09'00'81 \,\big|\, \underline{5009} \\ & \underline{25} \\ 10009 & \big|\; 90081 \\ & \big|\; \underline{90081} \end{array}$$

When one or more zeros occur at the end of the root, we have to note that there must be one for every two which occur at the end of the given number; thus, taking the case of $\sqrt{164025000000}$, we have—

$$\begin{array}{rl} & 16'40'25'00'00'00 \,\big|\, \underline{405000} \\ & \underline{16} \\ 805 & \big|\; 4025 \\ & \big|\; \underline{4025} \end{array}$$

177. When in the given number there are digits to the right of the units' mark, we may write the number in the

common fractional form and extract the root of numerator and denominator. The doing of this once, however, will show that it is really equivalent to going through the process of extraction with the number as given, and inserting the units' mark in the result at the stage where we begin to deal with the fractional digits. Thus—

$$\sqrt{1588.8196}=\sqrt{\frac{15888196}{10000}}$$
$$=\frac{\sqrt{15888196}}{\sqrt{10000}}=\frac{3986}{100}$$
$$=39.86\,;$$

or, we may simply figure as follows:—

$$\begin{array}{r|l}
 & 15'88.8196 \;|\; \underline{39.86} \\
 & \underline{9} \\
69 & 688 \\
 & \underline{621} \\
788 & 6781 \\
 & \underline{6304} \\
7966 & 47796 \\
 & \underline{47796}
\end{array}$$

the units' mark being inserted in the root at the time when we append the figures 81 to the remainder 67.

Exercises. Set CXI.

Find the second root of the following numbers:—

1. 506944.	2. 839056.	3. 889249.
4. 319225.	5. 454276.	6. 617796.
7. 788544.	8. 998001.	9. 826281.
10. 1317904.	11. 1555009.	12. 1811716.
13. 6723649.	14. 10857025.	15. 9455625.
16. 15147664.	17. 15896169.	18. 49112064.
19. 62362609.	20. 63824121.	21. 77404804.
22. 82446400.	23. 98604900.	24. 6412806400.
25. 689220009.	26. 612711009.	27. 2380952025.
28. 2689.4596.	29. 532316.16.	30. 658.076409.
31. .08708401.	32. .00576081.	33. .00088804.

34. $8\frac{867208}{1020100}$. 35. $6\frac{2110}{938961}$. 36. $4\frac{11867361}{111874816}$.
37. 81126049000000. 38. 108243216000o.
39. .000000160402653009. 40. .00008279196296227921.

178. We now come to consider the subject of the second root in connection with integral numbers such as 2, 3, 5, . . . , whose second roots are not integers; and the first important point to be noticed is, that not only are the second roots not whole numbers, but they are, further, not fractions expressible by a finite number of figures. For suppose that the second root of one of them can be expressed in the form of a fraction with a finite number of figures in the numerator and denominator, and that we have the fraction in its simplest possible form. Then, if this fraction were multiplied by itself, the result would necessarily be the original integer, whereas we know that if we multiply a fraction in its simplest form by itself, the numerator and denominator of the result contain the same factors as the original numerator and denominator, and that, consequently, the fraction cannot be simplified. Thus the second power of $\frac{15}{14}$ is $\frac{15 \times 15}{14 \times 14}$, and, the former numerator and denominator having no common factor, there cannot possibly be a factor common to the latter numerator and denominator.

A number, like 4, $\frac{9}{25}$, .01, . . . whose second root can be found and finitely expressed, is called an *exact second power;* any other number, such as 5, $\frac{1}{3}$, .1, . . . is called an *inexact second power*, and such numbers preceded by the sign of the second root, *e.g.* $\sqrt{5}$, $\sqrt{\frac{1}{3}}$, $\sqrt{.1}$, . . . are called *surd* expressions, or *surds.*

179. If an integral number be taken at random, and the process of extraction of the second root applied to it, we are enabled to say whether the number be an exact or inexact second power, and, if the former, what its second root is, if the latter, what is the largest integer whose second root is less than it. Thus, taking 81395, we have—

```
        8'13'95 | 285
        4
   48 | 413
      | 384
  565 | 2995
      | 2825
         170
```

and, consequently, know that 81395 is an inexact second power, and that the nearest second power below it is 170 less than it, that is to say, is 81225, the second root of which is 285.

180. Although there is no number, finitely expressible in the ordinary ways, whose second power is equal to any of the numbers 2, 3, 5, . . . , yet it is possible to find a number whose second power will *approximate* as nearly to any one of these numbers as may be desired. Thus, taking the case of 2, if we write it in the form $\frac{200}{100}$, and apply the process of the extraction of the second root to the numerator and denominator, we find the number $\frac{14}{10}$, whose second power is 1.96, that is to say, within less than a tenth of 2. Again, if we write it in the form $\frac{2000000}{1000000}$, and perform the like operations, we have the result 1.414, whose second power is 1.999396, that is to say, within less than a ten-thousandth of 2. Similarly, by using a fractional form with a still higher *even* power of 10 for denominator, a number may be found whose second power will be a still closer approximation to the number in question.

Numbers, like 1.4 and 1.414, whose second powers are approximations to a certain number, like 2, are viewed as *approximations to the second root* of that number. It is customary also to speak of the expression 1.414213 . . . , which consists of an infinite number of figures found in the manner just explained, as *the second root of* 2, and to say, for example, that 1.414 is the second root of 2 correct to three right-hand places, or to within less than a thousandth.

181. From what has been already said (§ 175), the learner will be prepared to find that in practice the approximations to the second root of 2 are not found from the fractional forms $\frac{141}{100}$, $\frac{14142}{10000}$, . . . , in the way explained, but simply as follows :—

$$\begin{array}{rl}
 & 2 \;|\; \underline{1.41....} \\
 & \underline{1} \\
24 & \underline{\left|\begin{array}{l}100 \\ 96\end{array}\right.} \\
281 & \underline{\left|\begin{array}{l}400 \\ 281\end{array}\right.} \\
 & 11900 \\
 & ....
\end{array}$$

where we start with the number 2, and apply the process, appending two zeros to each remainder when it is found.

Example. Find the second roots of 5.256 and .1 correct to within less than a thousandth.

$$\sqrt{5.256}=2.292... \qquad\qquad \sqrt{.1}=.316...$$

$$\begin{array}{rl}
 & 5.256 \;|\; \underline{2.292...} \\
 & \underline{4} \\
42 & \underline{\left|\begin{array}{l}125 \\ 84\end{array}\right.} \\
449 & \underline{\left|\begin{array}{l}4160 \\ 4041\end{array}\right.} \\
4582 & \underline{\left|\begin{array}{l}11900 \\ 9164\end{array}\right.} \\
 & 2736
\end{array}
\qquad\qquad
\begin{array}{rl}
 & .10 \;|\; \underline{.316...} \\
 & \underline{9} \\
61 & \underline{\left|\begin{array}{l}100 \\ 61\end{array}\right.} \\
626 & \underline{\left|\begin{array}{l}3900 \\ 3756\end{array}\right.} \\
 & 144
\end{array}$$

182. In the case of fractions whose numerator and denominator are inexact second powers, we may at once proceed to find approximations to the second roots of the numerator and denominator, and divide the one approximation by the other, thus—

$$\sqrt{\frac{2}{3}}=\frac{\sqrt{2}}{\sqrt{3}}=\frac{1.4142...}{1.7320...}=.816...,$$

or we may advantageously alter the form of the fraction so as to have either the numerator or denominator an exact second power, thus—

$$\sqrt{\frac{2}{3}}=\sqrt{\frac{2\times 2}{3\times 2}}=\frac{\sqrt{4}}{\sqrt{6}}=\frac{2}{2.4494\ldots}=.816\ldots,$$

or

$$\sqrt{\frac{2}{3}}=\sqrt{\frac{2\times 3}{3\times 3}}=\frac{\sqrt{6}}{3}=\frac{2.4494\ldots}{3}=.8164\ldots$$

Of these three modes the last is evidently the best, even in the case where the fraction as given has a numerator which is an exact second power. It may be improved upon by making the denominator an even power of 10, thus—

$$\sqrt{\frac{2}{3}}=\sqrt{.666666\ldots}=.816\ldots$$

EXERCISES. SET CXII.

Find the second root of the following numbers correct to three right-hand places :—

1. 5.	2. 17.	3. 18.	4. 101.
5. 67.	6. 217.	7. 346.	8. 404.
9. 1229.	10. 1407.	11. 1437.	12. 2117.

13. Find the whole number nearest to 5520150 which is an exact second power.

14. Find the second root of 2131 correct to within less than a hundred-thousandth, and thereby obtain $(46)^2$, $(461)^2$, $(4616)^2$, $(46162)^2$, $(461627)^2$, and $(4616275)^2$.

Find the second root of the following numbers correct to within less than a millionth :—

15. .1601.	16. .0801.	17. 16.08.	18. .002531.
19. 2.925.	20. 29.25.	21. .0013.	22. .00013.
23. $\frac{3}{16}$.	24. $\frac{9}{13}$.	25. $\frac{3}{5}$.	26. $\frac{5}{7}$.
27. $3\frac{1}{3}$.	28. $6\frac{1}{7}$.	29. $8\frac{3}{8}$.	30. $101\frac{8}{11}$.

Perform the operations indicated in the following expressions, giving the results correct to within less than a ten-thousandth :—

31. $(\sqrt{7}-\sqrt{3})(\sqrt{7}+\sqrt{3})$.	32. $\sqrt{2}\times\sqrt{3}-\sqrt{6}$.
33. $\sqrt{(\sqrt{2}+\sqrt{3})}$.	34. $\frac{\sqrt{6}-2}{\sqrt{6}+2}$.
35. $\sqrt{\left(\frac{\sqrt{.1}-\sqrt{.001}}{\sqrt{.01}-\sqrt{.0001}}\right)}$.	36. $\sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}$.

The Extraction of the Third Root.

183. The principles which underlie the process of the extraction of the third root of a number are exactly the same as those which have appeared in connection with the extraction of the second root. It thus happens that what has to be said regarding the former is closely similar to what has already been said regarding the latter. Indeed it would not be difficult, and the learner would find it exceedingly instructive, to re-examine each statement made on the subject of the extraction of the second root, and try to find the corresponding statement which holds good under the heading we have now reached. For this reason only the more important points will be noticed in what follows.

184. First of all we find the third powers of the first nine integers to be—

$$1, 8, 27, 64, 125, 216, 343, 512, 729,$$

from which we know and have to remember that—

$$\sqrt[3]{1}=1, \quad \sqrt[3]{8}=2, \quad \sqrt[3]{27}=3, \quad \&c.$$

Then we note such related facts as—

$$5^3=125, \quad 50^3=125000, \quad 500^3=125000000, \ \ldots\ldots ,$$

and thus learn that—

$$\sqrt[3]{216000}=60,$$
$$\sqrt[3]{729000000}=900,$$

and numberless other results.

Similarly we may observe that—

$$5^3=125, \quad .5^3=.125, \quad .05^3=.000125, \ \ldots\ldots ,$$

and hence infer that—

$$\sqrt[3]{.064}=.4,$$
$$\sqrt[3]{.000008}=.02,$$

and so on.

185. We next go on to examine the difference between the third power of a number and the third power of another

which exceeds the former by a certain amount, in order that, having got one approximation to the third root of a number, we may be enabled readily to proceed from it to a better. The theorem found is, *The third power of the sum of two numbers is greater than the third power of the first of them by thrice the second multiplied by the second power of the first, thrice the first multiplied by the second power of the second, and the third power of the second.* Thus—

$$(32+4)^3 = 32^3 + 3\times 32^2\times 4 + 3\times 32\times 4^2 + 4^3,$$
$$\text{and } (32+6)^3 = 32^3 + 3\times 32^2\times 6 + 3\times 32\times 6^2 + 6^3;$$

so that, knowing, for example, that—

$$\begin{aligned} 70^3 &= 343000, \\ 3\times 70^2\times 5 &= 73500, \\ 3\times 70\times 5^2 &= 5250, \\ \text{and} \quad 5^3 &= 125, \end{aligned}$$

we have by addition—

$$75^3 = 421875.$$

186. Lastly, we seek to know the connection between the number of digits in an integer and the number of digits in its third power, so that, knowing the latter, we may readily tell the former, and thus be enabled, when a number is given, to take the most suitable first approximation to its third root. The connection found is that *the number of digits in the third power of an integer is three times the number of digits in the integer itself, or is either one or two less than this.* Thus the number of digits in 76819825251 being 11, which is one less than three times four, there must be four digits in the third root; therefore, for a first approximation we have only to seek among the numbers 1000, 2000, 3000, ..., and the choice from these is easy.

187. Suppose now that we are asked to find the third root of the number 75686967. Beginning at the place of the units' mark we count the digits off in threes towards the left, and thus learn that the highest unit in the result is

the *hundred.* Trying 400, we find its third power to be 64′000′000; trying 500, we find its third power to be 125′000′000; and the given number being between these third powers, we take 400 as our first approximation to the root wanted, and subtracting 64000000 from 75686967, obtain the remainder 11686967. Knowing that this remainder must be the sum of 3 times 400^2 multiplied by the additional portion of the root wanted, 3 times 400 multiplied by the second power of the additional portion, and the third power of this portion, we say that 3 times 400 multiplied by the additional number must be less than 11686967, and that therefore the additional number itself must be less than $11686967 \div (3 \times 400^2)$, which is equal to 24.3.... Accordingly, taking 20 as the second portion of the root, we find thrice 20 times the second power of 400, thrice 400 times the second power of twenty, and the third power of 20, add the three results together and subtract the sum from 11686967. In the remainder, 1598967, we have now got the excess of the original number over the third power of 420, and therefore know that 1598967 must be greater than 3 times 420^2 multiplied by the third portion of the root, so that the said portion must be less than $1598967 \div (3 \times 420^2)$, which is equal to 3.0.... Trying 3 as the third portion, we proceed as before, and find there is no remainder—the root sought being thus 423.

The figuring of the process may be arranged as follows :—

$$
\begin{array}{rr|l}
 & 75'686'967 & \underline{400+20+3} \\
400^3 = & 64\,000\,000 & \\
\hline
3 \times 400^2 \times 20 = 9600000 & 11\,686\,967 & \\
3 \times 400 \times 20^2 = 480000 & & \\
20^3 = 8000 & & \\
\cline{1-1}
 & 10\,088\,000 & \\
\hline
3 \times 420^2 \times 3 = 1587600 & 1\,598\,967 & \\
3 \times 420 \times 3^2 = 11340 & & \\
3^3 = 27 & & \\
\cline{1-1}
 & 1\,598\,967 & \\
\hline
\end{array}
$$

Similarly, in the case of $\sqrt[3]{491169069}$, we have

$$
\begin{array}{rr|l}
 & & 491'169'069 \mid \underline{700+80+9} \\
700^3= & & \underline{343\,000\,000} \\
3\times 700^2\times 80 = & 117600000 & 148\,169\,069 \\
3\times 700\times 80^2 = & 13440000 & \\
80^3 = & \underline{512000} & \\
 & & 131\,552\,000 \\
\hline
3\times 780^2\times 9 = & 16426800 & 16\,617\,069 \\
3\times 780\times 9^2 = & 189540 & \\
9^3 = & \underline{729} & \\
 & & 16\,617\,069 \\
\hline
\end{array}
$$

188. The curtailment of figuring due to the omission of final zeros is readily made; that due to the substitution of one set of operations for another is not so simple. In the case of the first of the preceding examples, when we had found the second portion of the root, viz. 20, we performed the operations indicated in the expression

$$3\times 400^2\times 20 + 3\times 400\times 20^2 + 20^3,$$

and when we had found the third portion, viz. 3, the operations performed were those of a similar expression,

$$3\times 420^2\times 3 + 3\times 420\times 3^2 + 3^3.$$

Now instead of these we substitute

$$\{3\times 400^2 + (3\times 400 + 20)\times 20\}\times 20$$
$$\text{and } \{3\times 420^2 + (3\times 420 + 3)\times 3\}\times 3$$

respectively; that is to say, taking the former, we now multiply 400^2 by 3, and set aside the result; then multiply 400 by 3, add 20 to the product, and multiply the sum by 20; then adding this result to that formerly set aside, we multiply the whole by 20. The legitimacy of this depends upon the well-known fact that to multiply the sum of one or more numbers by another number we may multiply each

of the former numbers separately and add the results: thus, shortly,

$$\begin{aligned} &\{3\times 400^2 + (3\times 400+20)\times 20\,\}\times 20 \\ =&\{3\times 400^2 + \;\; 3\times 400\times 20 \;\; + 20^2\}\times 20 \\ =&\;\; 3\times 400^2\times 20 \;+\; 3\times 400\times 20^2 \;+\; 20^3. \end{aligned}$$

These changes, then, having been made, the figuring in the example referred to may stand as follows:—

$$\begin{array}{llr|l} & & 75'686'967 & \underline{423} \\ & & 64 & \\ \hline 122 & 48 & 11686 & \\ & \underline{244} & & \\ & 5044 & 10088 & \\ \hline 1263 & 5292 & 1598967 & \\ & \underline{3789} & & \\ & 532989 & 1598967 & \\ \hline \end{array}$$

Here we first find the greatest number whose third power is less than 75 to be 4, place the third power of the 4 below the 75, subtract, and append to the remainder the three figures 686 which follow the 75. Secondly, taking three times the 4, that is 12, and three times the second power of 4, that is 48, and placing them separately to the left of the 11686, we find that the 48 is contained 2 times in the 116; and annexing this 2 to the 12 on the left, we multiply the 122 thus got by the 2, put the result below the 48, but with two of the figures to the right of the 8, add, multiply the sum by the 2, place the product below the 11686, and subtract, thus getting the second remainder, 1598, to which we append the next three figures 967 of the original number. Thirdly, taking three times the 42 on the right of the original number, and three times the second power of the 42, viz. 5292, and placing them separately on the left of the 1598967, we inquire how often the 5292 is contained in the 15989; and so on, exactly as before.

Examples. Find the third root of 43095.8795 correct to within less than a hundredth, and the third root of .05 and of $\frac{8}{11}$ correct to within less than a thousandth.

(I.) $\sqrt[3]{43095.8795} = 35.06....$

		43'095.8795 | 35.06....
		27
95	27	16095
	475	
	3175	15875
10506	367500	220879500
	63036	
	36813036	220878216
		1284

(II.) $\sqrt[3]{.05}=.368....$

		·050 | .368....
		27
96	27	23000
	576	
	3276	19656
1088	3888	3344000
	8704	
	397504	3180032
		163968

(III.) $$\sqrt[3]{\frac{8}{11}}=\sqrt[3]{\frac{8\times 11^2}{11^3}}=\frac{\sqrt[3]{968}}{11}=\frac{9.892...}{11}=.899...$$

or— $$=\sqrt[3]{.727272727...}=.892...$$

Exercises. Set CXIII.

What is the third root of the following numbers?—

1. 1000, 8000, 64000.
2. 512000, 8000000000.
3. 27000000, 1000000000.
4. .001, .027, .000343.
5. .000001, .000000064.
6. .000000008, .000000216.
7. If a whole number of 7 digits be multiplied by a whole number of 6 digits, how many digits will there be in the result?
8. If a whole number of 8 digits be multiplied by a whole number of 9 digits, and the product be multiplied by a whole number of 10 digits, how many digits will there be in the final result?

Find the third root of the following numbers :—

9. 10648.	10. 32768.	11. 91125.	12. 262144.
13. 175616.	14. 636056.	15. 474552.	16. 970299.
17. .166375.	18. 438.976.	19. .024389.	20. .005832.

21. 12812904.	22. 88716536.	23. 306182024.
24. 688465387.	25. 938313739.	26. 1609840448.
27. .000313046839.	28. .000057960603.	29. .000009129329.
30. 3395290.527.	31. 9090072.503.	32. 216865.152512.
33. .275767330229.	34. 472183192.403.	35. 884.183155192.

Find the third root of the following numbers correct to within less than a thousandth :—

36. 3.	37. 6.	38. 30.	39. 47.
40. 111.	41. 194.	42. 215.	43. 705.
44. 2293.	45. 4724.	46. 5742.	47. 9139.

Find the third root of the following numbers correct to six right-hand places :—

48. .818.	49. .047.	50. .009.	51. .000349.
52. $.\dot{0}\dot{1}$.	53. $.0\dot{7}$.	54. $1\frac{1}{3}$.	55. $26\frac{1}{4}$.

Perform the operations indicated in the following expressions, giving the results correct to three right-hand places :—

56. $\sqrt[3]{(\sqrt[3]{.1}-\sqrt[3]{.0001})}$.	57. $2\times\sqrt[3]{3}+\sqrt[3]{24}$.
58. $\sqrt[3]{2}-\sqrt[3]{.2}-\sqrt[3]{.02}$.	59. $\sqrt[3]{5}\times\sqrt[3]{25}$.
60. $\sqrt[3]{5^2}+(\sqrt[3]{5})^2$.	61. $(\sqrt[3]{2}+\sqrt[3]{5})+\sqrt[3]{7}+\sqrt[3]{(\frac{2}{3}+7)}$.

Extraction of the Fourth and Higher Roots.

189. From the parallel study of the extraction of the second and third roots the possible generality of the method is brought strongly before the learner. On examination this possibility would be fully confirmed. It would be found that in the extraction of the *fourth* root we should mark off the digits of the given number in *fours*, find the first digit of the root sought from knowing the *fourth* powers of the first nine integers, and then repeatedly apply the theorem regarding the *fourth* power of the sum of two numbers; and similarly in other cases. As we advance to the higher roots, however, the fundamental theorem becomes more complicated, and the labour of extraction rapidly increases.

It can also be shown quite generally that *if an integral*

number be not a power of an integer, neither is it that power of any fraction finitely expressible. We thus have *exact* and *inexact* fourth powers, fifth powers, &c., and *surd* expressions, like $\sqrt[4]{4}$, $\sqrt[5]{6}$, &c.

190. It is important to notice, however, that the fourth, sixth, and other roots may also be found by means of the processes for the extraction of the second and third roots. In raising a number, 3 say, to the fourth power, we multiply 3 by 3 with the result 9, then, instead of multiplying this result by 3 and the new result again by 3, we may simply multiply the first result, viz. 9, by 9. Thus the fourth power is the second power of the second power, and therefore to find the fourth root of a number we have only to find its second root and the second root of the result. Again, since

$$\begin{aligned} 5^6 &= 5\times5\times5\times5\times5\times5 \\ &= 5^2\times5^2\times5^2 \qquad \text{or } = 5^3\times5^3 \\ &= (5^2)^3, \qquad\qquad\quad = (5^3)^2, \end{aligned}$$

it follows that

$$\begin{aligned} \sqrt[6]{15625} &= \sqrt[3]{\sqrt[2]{15625}} = \sqrt[3]{125} = 5, \\ \text{or } &= \sqrt[2]{\sqrt[3]{15625}} = \sqrt[2]{25} = 5. \end{aligned}$$

Similarly,

$$\sqrt[9]{7} = \sqrt[3]{\sqrt[3]{7}},$$
$$\sqrt[12]{7} = \sqrt[3]{\sqrt[2]{\sqrt[2]{7}}} = \sqrt[2]{\sqrt[3]{\sqrt[2]{7}}} = \sqrt[2]{\sqrt[2]{\sqrt[3]{7}}},$$

and so on.

Exercises. Set CXIV.

Find the fourth root of each of the following numbers correct to within less than a thousandth :—

1. .0004. 2. 04. 3. .004. 4. $3.\dot{9}$.

5. Which of the following are not surds ?—$\sqrt[3]{4}$, $\sqrt[4]{8}$, $\sqrt[5]{32}$, $\sqrt[3]{81}$, $\sqrt[4]{32}$, $\sqrt[3]{9}$, $\sqrt[6]{729}$, $\sqrt[6]{343}$, $\sqrt[6]{64}$.

Find the sixth root of

6. 17596287801000000. 7. .000000010779215329.

Find the eighth root of each of the following numbers correct to within less than a hundredth :—

8. 1851. 9. $3\frac{1}{2}$. 10. $2.\dot{0}\dot{6}$. 11. $28+16\sqrt{3}$.

Find the ninth root of

12. 118.587876497. 13. .0000003226876977779.

14. The area of a square field is 3 ac. 3 ro. 25 sq. po. Find the length of one side in links.

15. The bulk of a cube is 1 cub. ft. $732\frac{3}{8}$ cub. in. Find the length of the edge.

16. Name the roots between the twelfth and twenty-fifth which can be found by extracting second and third roots, and show the operations to be performed in each case.

17. A number multiplied by the tenth part of itself is $1\frac{188}{180}$. What is the number?

18. The side of a square is 4 ft. 8 in. What will be the length of a side of another which is ten times as great?

19. A rectangular bar of metal 13 ft. long, 3 in. broad, and $1\frac{1}{2}$ in. thick is changed into a cube. What will the length of an edge of the cube be?

20. The total area of the faces of a cube is 1 sq. ft. $40\frac{183}{200}$ sq. in. Find the length of one edge.

21. A certain number, the third part of it, and the fourth part of it are multiplied together, the result being .000000144. Find the number.

22. A sum of £1000 is placed in a bank where the interest is calculated yearly and added to the principal, and after 3 years the sum has increased to £1157 12s. 6d. What is the rate of interest per cent. per annum?

23. The area of a circular plot is 44 sq. ft. $2\frac{1}{4}$ sq. in. Find approximately its diameter.

24. Perform the operations indicated in the expression

$$\{(\sqrt{2}+\sqrt[4]{4}+\sqrt[6]{8})\times(\sqrt[3]{3}+\sqrt[6]{9}+\sqrt[9]{27})\}^2,$$

giving the result correct to within a thousandth.

25. A cubical cistern when full contains 83946.85 litres of water. Find its depth.

26. The bulk of a brass ball is 9 cub. in. Find approximately its radius.

27. A sum of £100000 lay in a bank for four years, the interest being added yearly to the principal. Knowing that the sum had thereby increased to 116985 17s. $1\frac{11}{25}$d., calculate the rate of interest per cent. per annum.

FRACTIONAL NUMBERS NOT FINITELY EXPRESSIBLE.

191. When only one unit, such as the *foot*, the *pound*, the *hour*, is employed in taking the measurement of any magnitude, and the unit is not contained in the magnitude a whole number of times exactly, the use of a fraction is necessary in order that a more accurate measurement may be expressed. For example, in measuring a line we may find it to be between 9 and 10 inches long, and if greater accuracy be wanted we may take a foot-rule subdivided into *sixteenths* of an inch, and find that the length is rather more than $9\frac{7}{16}$ inches. To have perfect accuracy, however, it would be necessary to find some aliquot part of the inch other than the sixteenth which would be contained a whole number of times exactly in the portion at first left unmeasured. Now the question here arises whether this be always possible. May there not be a line less than an inch long, such that, were we to try to measure it by the half, the third, the quarter, and all the other aliquot parts of the inch in succession, no one of them would be found contained a whole number of times in it exactly? The answer is that unquestionably there may.

It is known from geometry that the side and diagonal of a square are incommensurable—that is to say, have no common measure. Now if the lengths of these lines could be finitely expressed in fractional form, like $\frac{2}{3}$ and $\frac{15}{16}$, they *would* have a common measure, like $\frac{1}{48}$, *i.e.* $\frac{1}{3} \times \frac{1}{16}$; and therefore it follows that if either of them be finitely expressible in figures, the other is not. Thus, if a square be described whose side is exactly 1 ft. in length, then the length of the diagonal is between 1 ft. and 2 ft., more accurately between 1.414 ft. and 1.415 ft., but its exact excess over 1 ft. cannot be indicated by any fraction whatever with a finite denominator, and the attempt to do so results in the

interminate expression 1.4142..., which we have already referred to as being called *the second root of* 2, and which we for shortness write "$\sqrt{2}$."

Again, it is known that a diameter of a circle and the circumference are incommensurable, and thus it similarly happens that if a circle be drawn with a diameter exactly 1 in. long, the length of the circumference is somewhat more than 3 in., more nearly $3\frac{1}{7}$ in., more nearly $3\frac{16}{113}$ in., more nearly still 3.14159265 in., but not exactly expressible by means of any fraction whatever with a finite number of figures in the numerator and denominator. For the interminate expression in this case we write for shortness π.

192. The term *incommensurable*, which is only strictly applicable in speaking of *two* things, such as the diagonal and the side of a square in geometry, or the numbers $4\frac{1}{2}$ and $\sqrt[3]{5}$ in arithmetic, is also used absolutely* in the latter science to denote an interminate numerical expression of the kind just referred to. Thus $\sqrt[3]{5}$, apart from any mention of $4\frac{1}{2}$ or anything else, is called an *incommensurable number*, while $4\frac{1}{2}$, and numbers like it which can be expressed finitely, are called *commensurable;* and if it be asked, Incommensurable or commensurable with what? the answer is, With any number which can be expressed in the usual way with a finite number of figures.

193. In a magnitude such as a mathematical line there are no breaks, or, as the expression is, the magnitude is *continuous*. Numbers, on the other hand, are *discontinuous*, that is to say, we pass from one number to another by a series of leaps, small or large, equal or unequal; thus, 1, 2, 3, ..., or $1\frac{1}{100}$, $1\frac{1}{16}$, $1\frac{1}{10}$, ... It is because of this difference that it is possible to conceive of *finite magnitudes which are not finitely expressible in figures*, and it is the occurrence of such magnitudes and the attempt to express them in figures which leads to *incommensurable numbers*.

* Compare the two usages of the word *prime* (p. 92).

EXERCISES. SET CXV.

1. The interminate numerical expression $1.5\dot{3}\dot{4}$ is not an incommensurable number. What measure has it in common with 1? and what with $\frac{1}{4}$?

2. Which is the more general term, *incommensurable number* or *surd?* and why?

3. Give an example of two numbers which are separately incommensurable but relatively commensurable.

CONCLUSION.

194. Within the limits of an ordinary text-book it is only possible to present a simple outline of many of the subjects touched upon. For the reader who has advanced thus far, and who wishes to continue in any direction the study of the subject, a few notes are herewith added which may be useful to him as a guide at the outset.

195. Of general text-books there may be recommended those of Cornwall and Fitch,[1] Brook Smith,[2] and Voruz,[3] which approximate more or less closely in scope to the present work; and those of Sang[4] and Serret,[5] which deal less fully with the practical applications, but are valuable otherwise. Baltzer's[6] will exemplify a common mode of treating the subject in the secondary schools of Germany.

196. Of general historical writings the classical work is Dean Peacock's "History,"[7] given in the "Encyclopædia Metropolitana." There is also a store of material in De

(1) "The Science of Arithmetic." London.

(2) "Arithmetic in Theory and Practice." London.

(3) "Traité d'Arithmétique." Lausanne.

(4) "Elementary Arithmetic." Edinburgh, 1856. "The Higher Arithmetic." Edinburgh, 1857.

(5) "Eléments d'Arithmétique." Paris.

(6) "Elemente der Mathematik," b. i. Leipzig.

(7) Published also in the separate volume of the "Encyc. Metr.," entitled "Mathematical Treatises."

Morgan's "Catalogue of Arithmetical Books, from the Invention of Printing to the Present Time."(8) On special points of the history much research has been spent since Peacock wrote, but it would appear that the results have not as yet been collected and condensed.

197 [pp. 3—15]. The various systems of numerical Nomenclature and Notation afford one of the largest fields for historical inquiry. Under this head, beside the general history just referred to, the works of Cantor(9), Martin(10), Friedlein(11), Treutlein(12), Stoy(13), and Narducci(14) may be mentioned; but a large amount of all the information that has been printed is laid away in scientific magazines and the publications of learned societies.

Systems of notation on the Indo-Arabic principle with basis other than the base *ten* will be found treated of in Leslie's "Philosophy of Arithmetic"(15), a work likewise interesting from the historical point of view. The subject is also usually considered in modern English books on Algebra under the heading "Scales of Notation."

198 [pp. 15—38]. The performance of the Fundamental Operations by means of counters is very fully explained and illustrated by Leslie(15).

On the subject of extended Multiplication Tables, Leslie(15), Peacock(7), and De Morgan(8) may be consulted; the best and most readily obtained is Bremiker's edition of Crelle's(16).

(8) London, 1847.

(9) "Mathematische Beiträge zum Culturleben der Völker." Halle, 1863.

(10) "Les Signes Numéraux et l'Arithmétique chez les peuples de l'Antiquité," &c. Rome, 1864.

(11) "Die Zahlzeichen und das elementare Rechnen," u.s.w. Erlangen, 1869.

(12) "Geschichte unserer Zahlzeichen," u.s.w. Karlsruhe, 1875.

(13) "Zur Geschichte des Rechenunterrichts." Jena, 1876.

(14) "Intorno ad un Manoscritto della Bibl. Aless." ecc. Roma, 1877.

(15) Edinburgh, 1817.

(16) "Rechentafeln welche alles Multipliciren und Dividiren mit Zahlen unter Tausend ganz Ersparen," u.s.w. Berlin, 1864.

The abridgment of the process of Division is considered in several of the text-books above mentioned, and there is a separate work by Guy[17] which is well worth examination.

199 [pp. 38—45, etc.]. On the subject of Units of Measurement in general, and our own units in particular, the articles "Standard" and "Weights and Measures," in the "Penny Cyclopædia," and the writings referred to in these articles may be recommended. The most recent book is "The Science of Weighing and Measuring," by the present Warden of the Standards. Information as to the modern units of other countries may be found in the commercial books known as *Cambists* (see § 208), and in a handy form in Woolhouse's "Weights, Measures, and Money of all Nations."

200 [pp. 85—96]. The Prime Factors of all integers from 1 to 10,000 will be found in the first edition of Barlow's Tables[18]: the *lowest* prime factor of the same integers is also given under the article "Prime Numbers," in Hutton's Dictionary[19]; and Oakes's[20] table, the most recent English work of the kind, effects the same object for all integers between 1 and 100,000.

The most extensive factor tables, however, are German. Burckhardt[21] gave the lowest prime factor of each of the first three million integers, and Dase[22] did the same for the seventh, eighth, and ninth millions.

It is expected that the British Association will, at an early date, publish the tables dealing with the fourth, fifth, sixth, and tenth millions.

From all these tables lists of Prime Numbers may be easily made. Of special Tables of Primes the most exten-

(17) "La Division Abrégée." 1846.
(18) "New Mathematical Tables, containing," &c. London, 1814.
(19) "A Philosophical and Mathematical Dictionary." London, 1795.
(20) "Machine Tables for Determining Primes, &c." London, 1865.
(21) "Tables des Diviseurs," &c. Paris, 1814, 1816, 1817.
(22) "Factoren Tafeln," u.s.w. Hamburg, 1862, 1863, 1865.

sive published in this country are those of Barlow [18] and Rees,[23] the former reaching 100,103, the latter 217,219.

201 [pp. 96—119]. The subject of Continued Fractions is discussed at considerable length in Todhunter's Algebra and other similar text-books.

202 [pp. 127—150]. On the subject of the first use of the units' mark or decimal point, the introduction of which inaugurated one of the most important epochs in the history of Arithmetic, there are several writings to be noted besides the historical works already mentioned. These are, Mark Napier's "Memoirs of John Napier," [24] De Morgan's essay "On some Points in the History of Arithmetic," [25] and a special paper by J. W. L. Glaisher,[26] read before the British Association in 1873.

For giving the decimal fraction equivalent to a fraction in the ordinary notation, Tables of Reciprocals are published. Barlow's [18] or [27] and Oakes's,[28] which give the results correct to seven places, may be recommended. A table by Goodwyn [29] gives correct to eight places the decimal fractions corresponding not only to the reciprocals of integers, but to all fractions in the ordinary form, less than $\frac{88}{991}$, and having numerator and denominator not greater than 1000.

The same decimal fractions with their complete cycles of recurring figures may be obtained from another table of Goodwyn's; [30] Gauss [31] also gave the cycles for the

(23) "Cyclopædia, or Universal Dictionary," &c., in 39 vols. London, 1819.

(24) London, 1834.

(25) "Companion to the British Almanac." London, 1851.

(26) "On the Introduction of the Decimal Point into Arithmetic." Reprinted in "Nature," vol. viii. p. 515.

(27) "Tables of Squares, Cubes, Square Roots, Cube Roots, Reciprocals of all Integer Numbers up to 10,000." London, 1873.

(28) "Tables of the Reciprocals of Numbers from 1 to 100,000, with their Differences," &c. London, 1865.

(29) "A Tabular Series of Decimal Quotients," &c. London, 1823.

(30) "A Table of the Circles arising from the Division of a Unit or any other Whole Number by all the Integers from 1 to 1024," &c. London, 1823.

(31) Werke, b. ii. Göttingen, 1863.

reciprocals of all the Primes less than 1000. In regard to these cycles there are many interesting theorems, and the subject has been often studied and written on, but there is perhaps no work which contains a collection of the results.[32]

In calculations with numbers which only approximately represent the magnitudes of the things measured, it is important, especially in the case of decimal fractions, to know the relation between the degree of approximation of the result and the degrees of approximation of the numbers operated with. The subject is treated of in the two French text-books first referred to, and a special discussion of it by Vieille[33] has been considered worthy of mention.

203 [pp. 155—167]. In regard to the proposed Decimal Coinage, the Report (1853) of the House of Commons Committee may be consulted, and on the question of the advantages and disadvantages of decimal systems of units in general, there will be found large stores of information in the Reports of the various Commissions appointed at different times from 1816 to 1856 to consider the subject.

Additional details regarding the Metric System may be obtained in the Arithmetic of Voruz above referred to[3]; of separately published accounts, one of the latest, Dr. Barnard's,[34] may be mentioned. The great scientific work bearing on the subject is the "Base du Système Métrique Décimal."[35] The fullest information as to the progress of the system in recent years will be found in the "Procès Verbaux" of the International Metric Commission, or in the accounts of these given by the Warden of the Standards in his annual reports.[36] For help in making the change

(32) Of recent writings may be mentioned papers in the "Nouvelle Correspondance Mathématique," t. i. Bruxelles, 1875. "Messenger of Mathematics," vols. ii., iii., iv. Cambridge, 1873—75.

(33) "Théorie Générale des Approximations numériques."

(34) "The Metric System of Weights and Measures." New York, 1872.

(35) 4 vols. Paris, 1806—1821.

(36) For short notices of the Proceedings see "Nature," vols. vi., vii., viii., x., xi., xiii., xv.

from the French mode of expressing the measure of a magnitude to the British mode, tables such as Rutter's [37] are in use.

204 [pp. 171—274]. Of text-books explaining and illustrating the various kinds of calculations required in the practical affairs of life, there may be recommended Tate's "Counting-House Guide,[38] which deals with the less simple of the calculations of commerce, and Bleibtreu's "Political Arithmetic," [39] which is more occupied with the departments of calculation implied in the name. Text-books regarding single branches of business will be given in the order of the subjects.

205 [pp. 222—239]. For use in bank offices the results of calculations of interest are tabulated and published under the title of "Interest Tables." Gumersall's, King's, and Laurie's may be taken as examples.

A more complete discussion of the subject of Compound Interest than is given in the preceding, will be found in the special treatise of Thomson;[40] and of tables falling under this head there may be instanced those of Rance.

206 [pp. 239—249]. On the subjects with which discount is connected, viz. bills of exchange, promissory notes, &c., the treatise of Chitty [41] is considered a standard. From what has been before said, it will be understood that the tables in use by bill-discounters are Interest Tables.

207 [pp. 249—259]. Information in regard to the funded debt of our own country may be found in any good history of England, but Doubleday's "History" [42] may be recommended as dealing specially with the subject. As to the

(37) "Metrical System of Weights and Measures." London.

(38) London.

(39) "Politische Arithmetik." Leipzig.

(40) "Theory of Compound Interest." London.

(41) "A Treatise on Bills of Exchange, Promissory Notes," &c. London.

(42) "A Financial, Monetary, and Statistical History of England," &c. London.

amounts of the debts of the various states of the world, and other details connected therewith, a good book of reference is Fenn's "Compendium;[43] and as to the laws bearing on the subject there is a reference book by Royle.[44]

208 [pp. 259—267]. On the principles of International Exchange the popular works of Goschen[45] and Bagehot[46] may be read; and an excellent book for actual use in the counting-house is Tate's "Modern Cambist." Exchange Tables are published for almost every pair of trading nations in the world, so that no example need be given.

209 [pp. 274—286]. The finding of the Areas of Surfaces and the Bulks of Solids is the subject of books on *Mensuration* which are numerous. Todhunter's may be mentioned as an example of a good first book on the subject.

210. The more difficult calculations connected with the business of Insurance have not been considered in the present text-book. Two special works on the subject are De Morgan's "Essay on Probabilities,"[47] and Haberl's "Political Arithmetic."[48]

211 [pp. 286—312]. On the subject of Powers and Roots there is probably nothing better than Sang's "Higher Arithmetic,"[4] and the best book of tables is undoubtedly Barlow's,[27] above referred to. A general process for the extraction of roots is included in Horner's "Method of Solving Equations," an account of which will be found in Professor Thomson's Arithmetic,[49] and in most of the books dealing specially with the subject.[50] In De Mor-

[43] "Compendium of the English and Foreign Debts," &c. London.
[44] "Laws Relating to English and Foreign Funds," &c. London.
[45] "The Theory of the Foreign Exchanges." London.
[46] "Lombard Street: a Description of the Money Market." London.
[47] London, 1838.
[48] "Lehrbuch der Politischen Arithmetik." Wien, 1874.
[49] A new and important edition of his father's well-known book is looked for from Professor James Thomson, F.R.S.
[50] Young's "Theory and Solution of Algebraical Equations," London, 1835; or Todhunter's "Theory of Equations."

gan's Catalogue[a] (p. 89) there is a very full list of writings on the method.

212. Notwithstanding the improvements made in arithmetical methods, there are many pursuits in which the time and labour required for calculation extend to a serious amount, and consequently much ingenuity has been spent in endeavours to effect a saving. The two main lines followed in these endeavours have been—

(I.) The preserving of the results of calculations that have been made, classifying them and publishing them in tabulated form. Special *Tables* of this kind have been frequently referred to in the preceding paragraphs. There are, however, most important tables of very general application, called Tables of Logarithms, which have not yet been mentioned; an account of them is given in the work by Sang above referred to, and in many of the ordinary text-books of Algebra. A very extensive list of arithmetical tables, other than those which are strictly mercantile, is published in the Report of the British Association for 1873.

(II.) The devising of Arithmetical Instruments and Machines. Of these there is an important list in the valuable catalogue of the Special Loan Collection of Scientific Apparatus, and the handbook of the collection contains an article on the subject by Professor H. J. S. Smith. The calculating machine, or Arithmometer, of M. Thomas, is already in use in a number of public offices.

213. In these notices of books, works on Algebra have been more than once mentioned. This is because Algebra is that division of the Science of Number which immediately follows on Arithmetic. To it, therefore, as the natural continuation and development of his subject, the reader is now finally referred.

RESULTS OF THE EXERCISES.

SET I.

(1) Six hundred and ninety-nine. One thousand nine hundred and ninety-nine. Eighty-three thousand nine hundred and ninety-nine. Sixteen million nine hundred and ninety-nine thousand nine hundred and ninety-nine. Nine hundred and ninety-nine billion nine hundred and ninety-nine thousand nine hundred and ninety-nine million nine hundred and ninety-nine thousand nine hundred and ninety-nine.

(2) Dozen, gross, score.

(4) Twelve. Here *dozen* corresponds to *-ty* (or ten) in the decimal system, and *gross* to *hundred*, a gross being *twelve twelves*, just as a hundred is *ten tens*.

(5) One, two, three, ... , eleven, one dozen, one dozen and one, one dozen and two, ... , one dozen and eleven, two dozen, two dozen and one, ... , three dozen, ... , four dozen, ... , eleven dozen and eleven, a gross, one gross and one, one gross and two, ... , one gross and eleven, one gross one dozen, one gross one dozen and one, ... , one gross two dozen and ten.

SET II.

(1) Millions. Hundred thousand millions. Thousand billions. Ten quadrillions.

(2) Nine hundred and ninety-nine thousand nine hundred and ninety-nine. One million.

(3) Thirty-seven.

(4) 3ε, 904, 3642, τ001.

(5) One dozen and two. Two dozen. Two gross. Eight gross eight dozen and eight. Six dozen and five gross three dozen and seven. Eight dozen gross.

SET III.

(1) Nine hundred and forty-six thousand eight hundred and twelve.

Two hundred and seventeen thousand one hundred and eighteen. Twelve thousand two hundred and four. Ten thousand six hundred and ten. One thousand one hundred and eleven.

(2) Two hundred and forty thousand and twenty-four. Twelve thousand and eleven. One hundred thousand. Ten thousand and one. One hundred and ten thousand one hundred.

(3) One million ten thousand one hundred. Twenty million two hundred thousand and two. Six hundred and six million six hundred and sixty.

(4) Two thousand one hundred and sixty-three million eight hundred and fourteen thousand two hundred and sixteen. Three hundred and fifteen thousand two hundred and ten million seven hundred and twelve thousand two hundred and twenty-eight.

(5) Two hundred and one billion one hundred and twenty thousand and twelve million one hundred and two thousand two hundred and ten. Twelve billion one hundred and four thousand.

(6) One thousand and ten billion one thousand million one hundred thousand. One hundred billion ten thousand million one thousand.

(7) One billion. Fifty thousand and fifty billion five thousand five hundred million five hundred thousand and five.

(8) Seventy thousand billion ten million two thousand. Six hundred billion sixty million and six.

(9) Six hundred quadrillion four thousand billion twenty thousand.

(10) Five quintillion four hundred and seventeen thousand one hundred and sixteen quadrillion two hundred and eleven thousand four hundred and ten trillion ten thousand six hundred billion seven hundred and twelve thousand million ten thousand and one.

Set IV.

(1) 98457. 243648. 111212. (2) 210317. 10702. 300010.
(3) 100100. 10001. 300000. (4) 72348564. 318164352.
(5) 1111212310. 112211719200. (6) 300010002. 100000000040.
(7) 25008000000. 30000000400000.
(8) 809980098908890. 2020202002020200.
(9) 20000000000200000000.
200000020000000000000.
(10) 6000000000400000000000000000.
12000010000100001000100000.

Set V.

(1) 10676434. (2) 1518008. (3) 12314879.
(4) 10348085. (5) 2596397526. (6) 6765871922.
(7) 497555091. (8) 8333629034. (9) 1381518341.
(10) 982778. (11) 1826358265. (12) 15393558261210.

(13)

					Total.
					497375
					431676
					376753
					414904
Total . .	1179814	527057	2499	11338	1720708

(14) 293. (15) 23928. (16) 14428. (17) 55223.

Set VI.

(1) 5049207. (2) 4303692. (3) 156057.
(4) 1473439982. (5) 9990999. (6) 9999998999.
(7) 1899969989. (8) 90498980. (9) 199879979969.
(10) 998999999695. (11) 19099999009799570.
(12) 40608. 4860. (13) 755.

(14)

			Increase.	Decrease.
				65172
			32856	
			17856	
				1152
				38575
Total . . .	3798631	3744444		54187

(15)

Excess of !	Over			
	Liverpool.	Glasgow.	Birmingham.	Leeds.
Manchester. . .	10770	27019	160388	244963
Liverpool . . .		16249	149618	234193
Glasgow			133369	217944
Birmingham . .				84575

(16) 31210. (17) 31210. (18) 31210. (19) 31210.
(20) 29762.
(21) $(60+35+17)-(12+21)$. $(82-43)-(430-399)$.

Set VII.

(1) 1086783290. 1630174935. 2173566580.
(2) 659512695. 879350260. 1099187825.

(3) 1592069728. 1990087160. 2388104592.
(4) 1728400810. 2074080972. 2419761134.
(5) 5899174140. 6882369830. 7865565520.
(6) 1536762395. 1756299880. 1975837365.
(7) 1160079912. 1305089901. 290019978.
(8) 6657168546. 1479370788. 2219056182.

Set VIII.

(1) 31426800000. (2) 7710000000. (3) 52366664000.
(4) 144992000000. (5) 182760000. (6) 491076000000.
(7) 459333000000. (8) 7439040000000. (9) 320000000000.
(10) 466560000000000.

Set IX.

(1) 6769353. 10201701. (2) 17069220. 264201840.
(3) 43470375. 48248200. (4) 487676354. 245248099.
(5) 84512840368. (6) 26105701167000. (7) 2946372000.
(8) 46839408000. (9) 1670196080700000. (10) 17699147440000.
(11) 691337284158024. (12) 121932631112635269.

Set X.

(1) 1307674368000. (2) 1024. (3) 640000.
(4) £43155. (5) £2885. (6) 122880.
(7) £7800. (8) 630 ; 116. (9) 79800 ; 2850.
(10) 56400 ; 36120. (11) 2494. (12) 1000000 ; 1000000.
(13) 1000000 ; 1000000. (14) 53369 ; 101 ; 1225043.
(15) 5678992. (16) 83570675. (17) 1728000 ; 1728000.
(18) 17984996402880100. (19) 408040000.
(20) 3×10^2 ; $10 \times (2^2 + 3^2)$; $(1+2+3+4) \times (1 \times 2 \times 3 \times 4)$; $(7+2) \times (7-2)$.

Set XII.

(1) 17346282 ; 11564188 ; 8673141.
(2) 27982095 ; $20986571\frac{1}{2}$; 16789257.
(3) 12986580 ; 10389264 ; 8657720.
(4) 4318865 ; $3599054\frac{1}{6}$; $3084903\frac{4}{7}$.
(5) $12736666\frac{5}{6}$; 10917143 ; $9552500\frac{1}{8}$.
(6) $11208056\frac{2}{7}$; $9807049\frac{2}{8}$; $8717377\frac{1}{9}$.
(7) $4369526\frac{5}{8}$; $3884023\frac{6}{9}$; $3177837\frac{6}{11}$.
(8) $2381291\frac{2}{9}$; $1948329\frac{2}{11}$; $1785968\frac{5}{12}$.
(9) $19571\frac{4}{12}$; $24560\frac{8}{12}$.
(10) $1370933\frac{8}{12}$; $98344\frac{9}{12}$.

Set XIII.

(1) $307614\frac{14}{103}$; $277932\frac{8}{114}$.
(2) $47538\frac{10}{1013}$; $42729\frac{421}{1127}$.
(3) $204421\frac{13}{234}$; $195242\frac{227}{426}$.
(4) $33823\frac{1866}{2346}$; $22950\frac{1621}{3456}$.
(5) $13368\frac{736}{748}$; $12135\frac{760}{824}$.
(6) $4226\frac{6873}{9372}$; $8587\frac{1114}{4613}$.
(7) $40044\frac{701}{941}$; $60003\frac{221}{628}$.
(8) $20008\frac{2211}{2567}$; $30001\frac{1036}{1712}$.
(9) $50404\frac{381}{386}$; $39305\frac{360}{496}$.
(10) $10291\frac{4612}{4992}$; $7340\frac{4824}{8999}$.
(11) $42830\frac{833}{898}$; $38500\frac{878}{999}$.
(12) $1000080\frac{81}{9999}$; $1010080808\frac{81}{99}$.

Set XIV.

(1) 120084. (2) 999. (3) 339199. (4) 327680.
(5) 43 pounds. (6) 99. (7) 15 hundredweight.
(8) $1034827\frac{18}{29}$. (9) $235\frac{211}{346}$. (10) £505. (11) 9999.
(12) 10. (13) 36 ; 498. (14) 64 ; 1. (15) 1.
(16) 16384 ; 1. (17) $8^4 \div 2^6 = 2^6$. (18) 14.
(19) $100 \div 4 - 100 \div 5 = 100 \div 20$; $(12 \div 4) \div 3 = 1$.
(20) $(4 \times 5 \times 6 - 315 \div 9) \times (24 + 2^4)$.

Set XV.

(1) 148 far. ; 6540s.
(2) 1416d. ; 5250s.
(3) 1440 min. ; 12960 sec.
(4) 9688 lb. ; 79520 cwt.
(5) 2016 lb. ; 32256 oz.
(6) 904 fur. ; 6582 ft.
(7) 55044 ft. ; 4488 yd.
(8) 768 ro. ; 2448 sq. in.
(9) 756 cub. ft. ; 1306368 cub. in.
(10) 6992 qt. ; 13984 pt.
(11) 71520d. ; 144720 far.
(12) 171360d. ; 206400d. ; 334560d.
(13) 259200 sec. ; 2520 hr.
(14) 35840 lb. ; 152 st.
(15) 80000 po. ; 11220 ft.
(16) 203280 sq. yd. ; 24624 sq. in.
(17) 7604928 cub. in. ; 416 gills.
(18) 12224 gall. ; 4320 gall.
(19) 1140 fourp. ; 6384 threep.
(20) 5640 halfp. ; 5040 halfp.
(21) 63360 in. ; 4014489600 sq. in. ; 254358061056000 cub. in.
(22) 0 gr.
(23) 4 far. = 1d.
48 far. = 12d. = 1s.
960 far. = 240d. = 20s. = £1.
(24) 16 dr. = 1 oz.
256 dr. = 16 oz. = 1 lb.
7168 dr. = 448 oz. = 28 lb. = 1 qr.
28672 dr. = 1792 oz. = 112 lb. = 4 qr. = 1 cwt.
573440 dr. = 35840 oz. = 2240 lb. = 80 qr. = 20 cwt. = 1 ton.

Set XVI.

(1) 206d. ; 952d. ; 486d.
(2) 3552 far. ; 15649 far. ; 2882 far.
(3) 18371d. ; 80683 far.

(4) 27807 sec. ; 21641 min. ; 1816800 sec.
(5) 9968 lb. ; 31600 oz. ; 179661 oz.
(6) 622 in. ; 6820 yd. ; 22356 ft.
(7) 4005 sq. in. ; 880 sq. po. ; 135036 sq. ft.
(8) 449280 cub. in. ; 653512 cub. in.
(9) 1096 gall. ; 2625 gall. ; 516 gills.
(10) 1014 fourp. ; 1193 fourp.
(11) 841 sixp. ; 1303 halfp.
(12) 906½ st. ; 717 halfcr.
(13) 31556929½ sec.

SET XVII.

(1) 7932d. ; 192s.
(2) 74880d. ; 6240s. ; £312.
(3) 901 da. ; 16 hr.
(4) 370 cwt. ; 22 lb. Troy.
(5) 704 yd. ; 18 mi.
(6) 121 ac. ; 317 sq. ft.
(7) 99 cub. yd. ; 3976 gall.
(8) £157 12s. ; £8 15s. 4d. ; £74 0s. 8¾d.
(9) £11 17s. 6¼d. ; £33 3s. 5½d. ; £30 4s. 2¼d.
(10) 25 da. 6 hr. 24 min. 2 sec. ; 23 wk. 5 da. 16 hr. 10 min.
(11) 11 cwt. 2 lb. ; 1 ton 1 cwt. 10 lb. 12 oz.
(12) 56 tons 18 cwt. ; 3 lb. 8 oz. 7 dwt. 16 gr.
(13) 76 yd. 2 ft. 1 in. ; 6 fur. 38 po. 5 yd. 2 ft. ; 69 mi. 11 po. 2 yd. 9 in.
(14) 13 ac. 2 ro. 11 sq. po. ; 6 ac. 2 ro. 9 sq. po. 29¾ sq. yd. ; 1 ac. 3 sq. po. 3 sq. yd. 138 sq. in.
(15) 3 cub. yd. 5 cub. ft. 1706 cub. in. ; 392 gall. 2 qt. 1 pt. ; 1276 qr. 7 bus. 1 pk. 1 gall.
(16) 891 flor. ; 314 halfcr.
(17) £32025 ; 37800 guin.
(18) 15552 lb. avoir. ; 630000 lb. Troy.

SET XVIII.

(1) £22 4s. 1d.
(2) £183 7s. 7d.
(3) £216 10s. 9d.
(4) £14 18s. 1½d.
(5) £18 16s. 11d.
(6) £24 9s. 5d.
(7) £66 4s. 7¾d.
(8) £276 13s. 1¾d.
(9) £300 5s. 4½d.
(10) £288 16s. 11¾d.
(11) £328 15s. 8¼d.
(12) £433 19s. 3d.
(13) £285 17s. 1½d.
(14) £350 10s. 7d.
(15) £79 18s. 11¼d.
(16) £127021 16s. 6d.
(17) £159169 13s. 2¾d.
(18) £167015 16s. 2d.
(19) £131149 12s. 8½d.
(20) £635845 15s. 3d.
(21) £12875 7s. 8d.
(22) 2 lb. 3 oz. 3 dr.
(23) 60 lb. 13 oz. 5 dr.
(24) 56 lb. 12 oz. 13 dr.
(25) 2 cwt. 22 lb. 14 oz.
(26) 51 cwt. 2 qr. 13 lb.
(27) 52 cwt. 1 qr. 24 lb.
(28) 180 tons 8 cwt.
(29) 165 tons 12 cwt.
(30) 60 hr. 48 min. 53 sec.
(31) 49 hr. 20 min. 45 sec.

(32) 56 da. 8 hr. 58 min.
(33) 15 da. 4 hr. 24 min. 5 sec.
(34) 48 wk. 4 da. 20 hr.
(35) 40 yr. 179 da. 3 hr.
(36) 47 yd. 6 in.
(37) 58 yd. 1 ft. 1 in.
(38) 382 mi. 3 fur.
(39) 17 mi. 2 fur. 122 yd.
(40) 55 mi. 4 fur. 2 po.
(41) 12 mi. 4 fur. 3 po. 1½ yd.
(42) 126 gall. 1 qt. 1 pt.
(43) 159 gall. 2 qt. 1 pt.
(44) 104 qr. 5 bus. 2 pk.
(45) 190 qr. 5 bus.
(46) 59 bus. 1 pk. 1½ gall.
(47) 27 bus. 3 pk. 1 gall.
(48) 76 ac. 3 ro. 3 sq. po.
(49) 111 ac. 3 ro. 15 sq. po.
(50) 91 ac. 2 ro. 12 sq. po.
(51) 828 mi. 624 ac.
(52) 17 ac. 2 ro. 5 sq. po. 24 sq. yd.
(53) 14 sq. yd. 3 sq. ft. 86 sq. in.
(54) 31 sq. yd. 1 sq. ft. 19 sq. in.
(55) 103 cub. yd. 19 cub. ft. 601 cub. in.
(56) 121 cub. yd. 12 cub. ft. 799 cub. in.
(57) 88 cub. yd. 5 cub. ft. 734 cub. in.
(58) 1 lb. 8 oz. 5 dwt. 20 gr.
(59) 20 lb. 10 oz. 16 dwt.
(60) 30 lb. 4 oz. 4 dwt. 10 gr.

Set XIX.

(1) £83 9s. 5d.
(2) £179 5s. 6½d.
(3) £40 13s. 3d.
(4) £364 0s. 9d.
(5) £124 9s. 3d.
(6) £77 8s. 8d.
(7) £5 18s. 6¾d.
(8) £53 11s. 10¾d.
(9) £198 18s. 5¾d.
(10) £118 17s. 11½d.
(11) £16 19s. 11½d.
(12) £117 16s. 3¼d.
(13) 6 lb. 14 oz. 10 dr.
(14) 1 qr. 26 lb. 7 oz.
(15) 7 cwt. 1 qr. 21 lb.
(16) 6 cwt. 3 qr. 19 lb.
(17) 16 tons 18 cwt. 3 qr.
(18) 8 cwt. 1 qr. 19 lb.
(19) 3 wk. 2 da. 18 hr.
(20) 5 da. 20 hr. 50 min.
(21) 5 hr. 55 min. 46 sec.
(22) 4 hr. 47 min. 27 sec.
(23) 2 yr. 250 da. 21 hr.
(24) 19 hr. 52 min. 35 sec.
(25) 2 yd. 2 ft. 11 in.
(26) 8 yd. 1 ft. 7 in.
(27) 3 mi. 5 fur. 121 yd.
(28) 6 mi. 4 fur. 142 yd.
(29) 5 mi. 5 fur. 19 po.
(30) 1 mi. 7 fur. 32 po. 4½ yd.
(31) 8 gall. 2 qt. 1½ pt.
(32) 15 gall. 2 qt. ½ pt.
(33) 18 qr. 5 bus. 3 pk.
(34) 26 qr. 3 bus. 3 pk.
(35) 16 bus. 3 pk. ½ gall.
(36) 8 bus. 3 pk. 1 gall. 3 qt.
(37) 18 ac. 3 ro. 15 sq. po.
(38) 9 ac. 30 sq. po.
(39) 156 ac. 2 ro. 16 sq. po.
(40) 134 sq. yd. 8 sq. ft. 61 sq. in.
(41) 83 sq. yd. 4 sq. ft. 48 sq. in.*
(42) 5 ac. 3 ro. 22 sq. po. 22¼ sq. yd.
(43) 242 cub. yd. 24 cub. ft. 1632 cub. in.*

* The numbers of inches in Ex. 41 were meant to occupy the place of the corresponding numbers in Ex. 43, and *vice versa*, the numbers in the results then being 48 sq. in. and 1632 cub. in. respectively.

(44) 141 cub. yd. 10 cub. ft. 1063 cub. in.
(45) 346 cub. yd. 17 cub. ft. 1617 cub. in.
(46) 1 lb. 18 dwt. 17 gr. (47) 2 lb. 8 dwt. 8 gr.
(48) 4 oz. Troy 244 gr. (49) £335 6s. 5d.
(50) 2 mi. 834 yd. ; 1083 yd. ; 1186 yd.

SET XX.

(1) £27 12s. 6d. ; £41 8s. 9d.
(2) £83 15s. 0¾d. ; £111 13s. 5d.
(3) £659 14s. 2d. ; £824 12s. 8½d.
(4) £879 3s. 6½d. ; £1055 0s. 3d.
(5) £567 13s. 7½d. ; £662 5s. 10¾d.
(6) £1530 5s. 5½d. ; £1748 17s. 8d.
(7) £702 8s. 6d. ; £790 4s. 6¾d.
(8) £1960 5s. 3¾d. ; £2178 1s. 5½d.
(9) £3143 15s. 2½d. ; £2200 12s. 7¾d.
(10) £3975 3s. 9d. ; £3180 3s.
(11) £7649 3s. 1½d. ; £6884 4s. 9¾d.
(12) £3978 6s. 5½d. ; £4376 3s. 1½d.
(13) £1035 19s. 10½d. ; £1130 3s. 6d.
(14) £818 5s. 3d. ; £681 17s. 8½d.
(15) 22 tons 5 cwt. 2 qr. ; 25 tons 19 cwt. 3 qr.
(16) 28 cwt. 9 lb. ; 50 cwt. 2 qr. 5 lb.
(17) 134 yd. 2 ft. 8 in. ; 168 yd. 1 ft. 10 in.
(18) 142 ac. 17 sq. po. ; 129 ac. 30 sq. po.
(19) 43 gall. 2 qt. ; 72 gall. 2 qt.
(20) 139 sq. yd. 8 sq. ft. 74 sq. in. ; 599 sq. yd. 6 sq. ft. 132 sq. in.

SET XXI.

(1) £297 8s. 4d. (2) £1740 2s. 7½d. (3) £867 11s. 11¼d.
(4) £2233 7s. 9¾d. (5) £979 19s. 0¾d. (6) £3737 18s. 1½d.
(7) £5658 12s. 3¾d. (8) £2579 6s. 2d. (9) £2429 12s. 1d.
(10) £5729 19s. 11d. (11) £24438 2s. 4½d. (12) £2426 0s. 9¾d.
(13) £14089 6s. 8¼d. (14) £600 13s. 8½d. (15) £599 3s. 6d.
(16) £1505 5s. 1d.
(17) 3 tons 17 cwt. 3 qr. 24 lb. 13 oz. ; 6 tons 6 cwt. 2 qr. 7 lb.
(18) 640 ac. 2 ro. ; 407 ac. 1 ro. 24 sq. po.
(19) 21 da. 15 hr. 42 min. 9 sec. ; 29 da. 15 hr. 21 min. 57 sec.
(20) 13357 mi. 1012 yd. ; 6436 mi. 943 yd.

SET XXII.

(1) £18 13s. 8d. (2) £72 8s. 9d. (3) £64 9s.
(4) £263 0s. 2½d. (5) £528 13s. 6d. (6) £1063 4s. 3d.

(7) £393 4s. 7d. (8) £1953 2s. 6d. (9) £2047 6s. 8d.
(10) £16546 5s.
(11) 152 tons 18 cwt. ; 166 tons 16 cwt.
(12) 668 yd. 2 ft. 7 in. ; 796 yd.
(13) 5791 hr. 52 min.
(14) 3402 qr.
(15) 6932 ac. 3 ro. 10 sq. po. ; 4067 ac. 1 ro.
(16) 2598 cub. yd. 11 cub. ft. 64 cub. in.

SET XXIII.

(1) £105 5s. $6\frac{1}{4}$d. (2) £1331 17s. $11\frac{1}{4}$d. (3) £681 15s. $10\frac{1}{2}$d.
(4) £99 6s. $10\frac{1}{2}$d. (5) £42 0s. 8d. (6) £214 19s. $10\frac{1}{2}$d.
(7) £311 15s. $9\frac{1}{2}$d. (8) 807 da. 18 hr. 30 min.
(9) 488 oz. 12 dwt. 18 gr. (10) 470 ac. 1 ro. 34 sq. po.
(11) 25 cub. yd. 15 cub. ft. 626 cub. in.
(12) 28 tons 3 cwt. 2 qr. 2 lb.

SET XXIV.

(1) £837 19s. 9d. (2) £254 16s. $5\frac{3}{4}$d. (3) £2131 2s. $10\frac{1}{2}$d.
(4) £10401 6s. 7d. (5) £13603 5s. $7\frac{1}{2}$d. (6) £8289 14s. $10\frac{1}{4}$d.
(7) £26675 8s. 6d. (8) £24380 5s. $3\frac{3}{4}$d.
(9) 2478 qr. 3 bus. 2 pk. (10) 37171 yd. 9 in.

SET XXV.

(1) £49 1s. 9d. ; £32 14s. 6d.
(2) £139 4s. 10d. ; £104 8s. $7\frac{1}{2}$d.
(3) £158 19s. $7\frac{1}{4}$d.$\frac{1}{4}$ * ; £127 3s. $8\frac{1}{4}$d.
(4) £153 5s. $0\frac{1}{4}$d.$\frac{1}{8}$; £127 14s. $2\frac{1}{4}$d.
(5) £52 6s. $8\frac{3}{4}$d.$\frac{1}{8}$; £44 17s. $1\frac{3}{4}$d.
(6) £29 6s. 10d. ; £25 13s. $5\frac{3}{4}$d.
(7) £19 19s. $2\frac{3}{4}$d.$\frac{3}{8}$; £17 14s. $10\frac{1}{2}$d.
(8) £77 8s. 9d. ; £69 13s. $10\frac{1}{2}$d.
(9) £14 0s. $0\frac{1}{4}$d. ; £12 14s. $6\frac{3}{4}$d.$\frac{1}{11}$
(10) 16s. $4\frac{1}{4}$d.$\frac{6}{11}$; 15s.
(11) 58 hr. 2 min. 25 sec. ; 26 hr. 22 min. 55 sec.
(12) 9 cwt. 3 qr. 21 lb. ; 8 cwt. 3 qr. $21\frac{7}{10}$ lb.
(13) 3 yd. 1 ft. 7 in. ; 2 yd. 1 ft. $0\frac{8}{12}$ in.
(14) 1 ac. 3 ro. 25 sq. po. ; 1 ac. 2 ro. $39\frac{7}{12}$ sq. po.

* The $\frac{1}{4}$ here following the *d.* is understood to denote the fourth part of a *farthing*, not of a *penny*. A better mode of writing such results is afterwards given.

Set XXVI.

(1) £77 14s. $0\frac{1}{4}$d. (2) 13s. $9\frac{1}{4}$d. (3) £5 6s. $10\frac{1}{2}$d.
(4) £18 15s. $2\frac{1}{2}$d. (5) £19 16s. $2\frac{1}{4}$d. (6) £15 17s. $4\frac{3}{4}$d.
(7) 17s. $8\frac{3}{4}$d. (8) £12 0s. $5\frac{1}{4}$d. (9) £73 0s. $0\frac{1}{2}$d.
(10) 5s. $0\frac{3}{4}$d. (11) 6 cwt. 1 qr. 13 lb. ; 2 cwt. 2 qr. $16\frac{82}{113}$ lb.
(12) 7 lb. 13 oz. 12 dr. (13) 1 hr. 13 min. 47 sec. ; 56 min. $29\frac{64}{125}$ sec.
(14) 1 da. 13 min. (15) 3 po. 2 yd. 1 ft. ; 2 po. 4 yd. 1 ft. $8\frac{8}{89}$ in.
(16) 1 mi. 38 yd. ; 1154 yd. 1 ft. $8\frac{148}{299}$ in.
(17) 3 ro. 27 sq. po. ; 1 ac. 1 ro. $18\frac{118}{393}$ sq. po.
(18) 9 sq. yd. 67 sq. in.
(19) 2 bus. 1 gall. ; 5 bus. $1\frac{48}{79}$ gall.
(20) 17 cub. ft. 342 cub. in.

Set XXVII.

(1) £77 14s. $0\frac{1}{4}$d. (2) £13 19s. $4\frac{1}{2}$d.
(3) £5 6s. $10\frac{1}{2}$d. (4) £13 19s. $4\frac{1}{2}$d.
(5) 13s. $9\frac{1}{4}$d. (6) £2 0s. $0\frac{1}{4}$d.
(7) 1 cwt. 3 qr. 17 lb. (8) 1 bus. 3 pk. 1 gall. 3 qt.
(9) £3 17s. $2\frac{1}{4}$d. (10) £2 19s. $0\frac{1}{4}$d.
(11) 6 cwt. 1 qr. 13 lb. (12) 7 lb. 13 oz. 12 dr.
(13) 1 hr. 13 min. 47 sec. (14) 3 po. 2 yd. 1 ft.
(15) 9 sq. yd. 67 sq. in. (16) 17 cub. ft. 342 cub. in.

Set XXVIII.

(1) 3. (2) 1305. (3) 12. (4) 123. (5) 261.
(6) 41. (7) 315. (8) 105. (9) 98. (10) 57.
(11) 171.
(12) $59\frac{11587}{11781}$. If 524 be put in the exercise for 542 the result is 58.
(13) 67. (14) 388.

Set XXIX.

(1) £40821 10s. ; £1069. (2) £3461 5s. ; £1680 8s.
(3) £1536 4s. ; £6841 14s. (4) £1985 6s. ; £3 10s.
(5) £927 ; £12 13s. 4d. (6) £2697 ; £5 6s. 8d.
(7) £10620 6s. 8d. ; £2 1s. 8d. (8) £895 18s. 4d. ; £108 7s.

Set XXX.

(1) £234 7s. 6d. (2) £71 12s. 6d. (3) £208.
(4) £49 17s. 4d. (5) £32 18s. 4d. (6) £366 6s.
(7) £32 7s. 6d. (8) £259 0s. 6d. (9) £662 15s.
(10) £35 5s. 4d. (11) £4 15s. $7\frac{1}{2}$d. (12) £24 19s. $4\frac{1}{2}$d.
(13) £12 1s. $0\frac{3}{4}$d. (14) £28 14s. $8\frac{1}{4}$d. (15) £85 13s. $6\frac{3}{4}$d.

(16) £106 19s. $1\frac{1}{2}$d. (17) £23 11s. $10\frac{1}{4}$d. (18) £191 0s. $11\frac{3}{4}$d.
(19) £58 12s. $9\frac{3}{4}$d. (20) £125 0s. $3\frac{3}{4}$d.

SET XXXI.

(1) £413 5s. (2) £936 12s. 6d. (3) £405 6s. 8d.
(4) £1122 18s. (5) £2009 11s. 8d. (6) £1028 2s. 6d.
(7) £257 19s. (8) £850 12s. 4d. (9) £5975 15s. 8d.
(10) £3246 15s. 9d. (11) £436 5s. $7\frac{1}{2}$d. (12) £1448 14s. $0\frac{1}{4}$d.
(13) £1016 15s. $5\frac{1}{4}$d. (14) £2956 16s. $5\frac{1}{2}$d. (15) £1120 3s. $6\frac{1}{4}$d.
(16) £1395 18s. 2d. (17) £2454 10s. $8\frac{1}{2}$d. (18) £2561 5s. $1\frac{1}{4}$d.
(19) £3256 10s. 3d. (20) £17075 12s. $1\frac{3}{4}$d.

SET XXXII.

(1) £344 15s. (2) £180 16s. 8d. (3) £367 11s. 6d.
(4) £365 15s. (5) £170 10s. (6) £158 6s. 7d.
(7) £1166 2s. (8) £6260 12s. 6d. (9) £71395 10s. 9d.
(10) £18729 13s. 9d.

SET XXXIII.

(1) £2 17s. $0\frac{1}{2}$d. (2) £116 13s. 4d. (3) 115 gall. 2 qt.
(4) 19 tons 11 cwt. (5) £28 17s. 6d. (6) 59 ac. 1 ro. 6 sq. po.
(7) 5 hr. 50 min. (8) £1 7s. 6d. (9) £29 5s.
(10) 13 hr. 30 min.

SET XXXIV.

(1) £57 17s. 4d. (2) £412 2s. 6d.
(3) £5 7s. $8\frac{1}{2}$d. (4) £138 16s. 8d.
(5) 25777 sq. yd. 7 sq. ft. (6) £8 9s. 7d.
(7) 216 gall. 1 qt. (8) 715 ac. 1 ro. 10 sq. po.
(9) 177 hr. 17 min. 30 sec. (10) £56 10s. $2\frac{1}{2}$d.

SET XXXV.

(1) £13 6s. 4d. (2) £4 4s. (3) 228 mi.
(4) 5 tons 2 cwt. (5) 3 hr. 54 min. 3 sec. (6) 2 ac. 2 ro.
(7) 77 lb. 8 oz. (8) £344 2s. 6d. (9) 21 ft.
(10) £6 11s. $11\frac{3}{4}$d. (11) £264 11s. $2\frac{1}{2}$d.
(12) £108 15s. $8\frac{1}{2}$d. $\frac{113}{385}$ far.

SET XXXVI.

(1) £309 0s. 6d. (2) £355 10s. $5\frac{1}{4}$d. (3) £64 16s. $6\frac{1}{2}$d.
(4) £1000. (5) £57 3s. 4d. (6) 429 mi.
(7) 8 da.

Set XXXVII.

(1) 1131. (2) 16. (3) 13 lb. 14 oz.
(4) 222. (5) 2s. $0\frac{3}{4}$d. (6) £776 12s. 6d.
(7) £32 4s. 10d. (8) £10 13s. $4\frac{1}{2}$d. (9) £96.
(10) £25 11s. 8d. (11) £412 10s. (12) $5\frac{1}{2}$d.
(13) 127 ac. 3 ro. 29 sq. po. $12\frac{3}{4}$ sq. yd. (14) £562 2s. 6d., £843 3s. 9d.
(15) £68 18s. $1\frac{1}{2}$d. (16) £8 14s. $7\frac{1}{2}$d. (17) £14 4s. $5\frac{1}{4}$d.
(18) $1\frac{1}{4}$d. (19) 963, 1053. (20) 45.
(21) 13 lb. (22) 75025. (23) £7 13s.
(24) 10 mi. 1440 yd. (25) 11200. (26) £2 4s. 6d.
(27) 315. (28) $111078\frac{2}{9}$. (29) £12495 16s. $1\frac{1}{2}$d.
(30) 1601. (31) 8 ac. 1 ro. $36\frac{17}{40}$ sq. po.
(32) 8100. (33) 29. (34) 5 gall. 2 qt. 1 pt.
(35) 308. (36) £19 19s. 6d. (37) 79900.
(38) 1 hr. 24 min. $55\frac{2}{7}$ sec. (39) £250.
(40) 239445. (41) 272 ac. 2 ro. 27 sq. po., 240 ac. 37 sq. po.
(42) £1 3s. 4d. (43) 3705860. (44) 11s. 8d.
(45) 12279 cub. in. (46) 5s. 6d. (47) 126.
(48) $1\frac{1}{2}$d. (49) $\frac{1}{2}$d. (50) £5 13s. $5\frac{3}{4}$d.
(51) 26 lb. (52) 76 bus. (53) 192, 3756.
(54) 720. (55) 8 hr. (56) 600.
(57) 50456. (58) £9 17s. (59) £38 11s. 4d.
(60) £2 0s. $0\frac{1}{2}$d., £1 2s. 6d.

Set XXXVIII.

(1) $2\times2\times5\times5$; $2\times2\times2\times2\times2$; $2\times2\times2\times3\times3$.
(2) $2\times2\times2\times2\times3\times3$; $3\times3\times7$; $2\times2\times3\times3\times5$.
(3) $3\times5\times5$; $5\times5\times5$; $5\times5\times7$.
(4) $3\times3\times13$; $3\times3\times17$; $3\times3\times23$.
(5) 3×19 ; 3×29 ; $3\times7\times7$.
(6) $3\times5\times17$; $3\times3\times7\times7$; $3\times5\times37$.
(7) $3\times3\times5\times17$; $2\times2\times2\times3\times47$; $2\times2\times3\times43$.
(8) $2\times11\times17$; 11×47 ; $5\times11\times19$.
(9) $2\times2\times2\times3\times3\times3\times11$; $3\times3\times5\times11\times17$.
(10) $3\times3\times3\times3\times3\times11\times29$; $3\times3\times11\times11\times11\times23$.
(11) 13×29 ; 13×17 ; 17×19.
(12) $7\times19\times19$; $7\times13\times13$; $13\times23\times23$.
(13) 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.
(14) 163, 167, 173, 179.

Set XXXIX.

(1) 6. (2) 7. (3) 75. (4) 3. (5) 3.
(6) 4. (7) 72. (8) 2. (9) 27. (10) 99.

SET XL.

(1) 1. (2) 19. (3) 7. (4) 11. (5) 7.
(6) 7. (7) 13. (8) 1. (9) 2. (10) 23.

SET XLI.

(1) 221. (2) 14. (3) 19. (4) 39. (5) 26. (6) 17.

SET XLII.

(1) 60. (2) 120. (3) 120. (4) 1680.
(5) 180. (6) 360. (7) 3780. (8) 1440.
(9) 15600. (10) 3528. (11) 11400. (12) 3780.

SET XLIII.

(1) 2520. (2) 2520. (3) 5040.
(4) 1716. (5) 285285. (6) 109512.
(7) 26455. (8) 33565. (9) 77763.
(10) 106913.

SET XLIV.

(1) $2\frac{1}{2}$ ft. ; $1\frac{1}{12}$d. ; £$1\frac{3}{8}$.
(2) Seven-tenths, $\frac{7}{10}$.
(3) $\frac{60}{100}$, $\frac{101}{1000}$, $100\frac{1}{1000}$, $\frac{13}{100000000}$, $\frac{21}{1000000000}$.
(4) $\frac{25}{5}$, $\frac{50}{10}$, $\frac{500}{100}$, $\frac{5000}{1000}$.
(5) $\frac{8}{4}$, $\frac{28}{4}$, $\frac{52}{4}$, $\frac{80}{4}$; $\frac{96}{48}$, $\frac{336}{48}$, $\frac{624}{48}$, $\frac{960}{48}$.
(6) $\frac{3}{1}$, $\frac{13}{1}$, $\frac{23}{1}$, $\frac{103}{1}$; $\frac{432}{144}$, $\frac{1872}{144}$, $\frac{3312}{144}$, $\frac{14832}{144}$.
(7) $\frac{25}{8}$, $\frac{241}{8}$, $\frac{14}{5}$, $\frac{1004}{5}$, $\frac{30014}{15}$.
(8) $\frac{21}{10}$, $\frac{1433}{100}$, $\frac{17841}{1000}$, $\frac{10082619}{10000}$.
(9) $\frac{1703}{100}$, $\frac{18003}{1000}$, $\frac{240027}{10000}$, $\frac{340000013}{10000}$.
(10) $\frac{2672}{99}$, $\frac{2645}{98}$, $\frac{22400}{199}$, $\frac{347000}{999}$, $\frac{10230}{101}$.

SET XLV.

(1) $1\frac{1}{2}$, $3\frac{1}{3}$, 1, $2\frac{3}{8}$, $2\frac{7}{13}$, $1\frac{16}{39}$, 7.
(2) $11\frac{3}{10}$, $6\frac{67}{100}$, $41\frac{769}{1000}$, $30\frac{7}{1000}$, $11\frac{1}{81}$, $5\frac{4}{99}$.
(3) $35\frac{1}{7}$, $296\frac{9}{11}$, $3132\frac{4}{9}$, 4, 9, $280\frac{19}{80}$.
(4) 4, $9\frac{4}{5}$, 200, 8, $7\frac{7}{18}$, $246\frac{15}{24}$.

SET XLVI.

(1) Seventeen hundredths, $\frac{17}{100}$.
(2) $\frac{6}{9}$, $\frac{8}{12}$, $\frac{18}{27}$, $\frac{200}{300}$.
(3) $\frac{12}{24}$, $\frac{18}{24}$, $\frac{15}{24}$, $\frac{14}{24}$.
(4) $\frac{20}{30}$, $\frac{18}{30}$, $\frac{5}{30}$, $\frac{21}{30}$, $\frac{4}{30}$.
(5) $\frac{30}{70}$, $\frac{42}{70}$, $\frac{56}{70}$, $\frac{24}{70}$, $\frac{35}{70}$.
(6) $\frac{9}{15}$, $\frac{15}{25}$, $\frac{72}{120}$.
(7) $\frac{64}{189}$, $\frac{64}{144}$, $\frac{64}{99}$, $\frac{64}{80}$.
(8) $\frac{2}{10}$, $\frac{75}{100}$, $\frac{8}{10}$, $\frac{625}{1000}$, $\frac{28}{100}$, $\frac{1875}{10000}$.

SET XLVII.

(1) $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{7}$. (2) $\frac{19}{24}$, $\frac{47}{84}$, $\frac{28}{35}$. (3) $\frac{13}{18}$, $\frac{13}{14}$, $\frac{13}{15}$.
(4) $\frac{11}{27}$, $\frac{2}{5}$, $\frac{29}{35}$. (5) $\frac{7}{17}$, $\frac{16}{19}$, $\frac{9}{11}$. (6) $\frac{2}{3}$, $\frac{7}{8}$, $\frac{9}{10}$.
(7) $\frac{216}{395}$, $\frac{216}{313}$, $\frac{187}{203}$. (8) $\frac{287}{31426}$, $\frac{661}{1314}$, $\frac{5}{7}$.

SET XLVIII.

(1) $\frac{18}{24}$, $\frac{12}{24}$, $\frac{15}{24}$, $\frac{14}{24}$.
(2) $\frac{18}{30}$, $\frac{4}{30}$, $\frac{21}{30}$, $\frac{26}{30}$.
(3) $\frac{90}{240}$, $\frac{100}{240}$, $\frac{105}{240}$, $\frac{132}{240}$.
(4) $\frac{1680}{2940}$, $\frac{1120}{2940}$, $\frac{756}{2940}$, $\frac{1365}{2940}$, $\frac{240}{2940}$.
(5) $\frac{120}{660}$, $\frac{308}{660}$, $\frac{96}{660}$, $\frac{75}{660}$, $\frac{99}{660}$.
(6) $\frac{135}{630}$, $\frac{175}{630}$, $\frac{80}{630}$, $\frac{196}{630}$, $\frac{522}{630}$.
(7) $\frac{1560}{2925}$, $\frac{300}{2925}$, $\frac{810}{2925}$, $\frac{2275}{2925}$, $\frac{1053}{2925}$.
(8) $\frac{630}{2160}$, $\frac{1224}{2160}$, $\frac{384}{2160}$, $\frac{680}{2160}$, $\frac{855}{2160}$.
(9) $\frac{156}{324}$, $\frac{99}{324}$, $\frac{297}{324}$, $\frac{165}{324}$, $\frac{8}{324}$.
(10) $\frac{1650}{5610}$, $\frac{120}{5610}$, $\frac{1485}{5610}$, $\frac{3498}{5610}$, $\frac{22}{5610}$, $\frac{220}{5610}$.
(11) $\frac{720}{3420}$, $\frac{2280}{3420}$, $\frac{620}{3420}$, $\frac{380}{3420}$, $\frac{495}{3420}$, $\frac{304}{3420}$.
(12) $\frac{6}{11}$, $\frac{24}{35}$, $\frac{5}{7}$, $\frac{3}{5}$.
(13) $\frac{9}{35}$, $\frac{13}{49}$, $\frac{3}{11}$, $\frac{2}{7}$.
(14) $\frac{190}{231}$, $\frac{13}{30}$, $\frac{167}{385}$, $\frac{67}{154}$.
(15) $2\frac{6}{11}$, $\frac{57}{23}$, $\frac{632}{253}$, $\frac{81}{88}$.

SET XLIX.

(1) 2. (2) $2\frac{1}{18}$. (3) $1\frac{47}{60}$. (4) $\frac{6141}{10000}$.
(5) $999\frac{777}{1000}$. (6) $1\frac{73}{90}$. (7) $1\frac{37}{42}$. (8) $1\frac{79}{84}$.
(9) $9\frac{10}{11}$. (10) $19\frac{3}{40}$. (11) $134\frac{5}{216}$. (12) $39\frac{1}{3}$.
(13) $104\frac{1}{18}$. (14) $16\frac{73}{126}$. (15) $14\frac{13}{45}$. (16) $8\frac{1}{49}$.
(17) 20. (18) $14\frac{5}{78}$. (19) $5\frac{851}{1296}$. (20) $6\frac{9943}{10000}$.

SET L.

(1) $\frac{8}{19}$, $\frac{1}{3}$, $\frac{3}{16}$, $\frac{1}{2}$. (2) $1\frac{1}{5}$, $4\frac{1}{25}$. $3\frac{69}{1000}$, $15\frac{1}{4}$.
(3) $\frac{11}{35}$, $\frac{2}{45}$, $\frac{104}{245}$, $\frac{1}{195}$. (4) $\frac{73}{864}$, $\frac{73}{2125}$, $\frac{184}{2275}$, $\frac{77}{8640}$.
(5) $4\frac{1}{2}$, $8\frac{3}{7}$, $25\frac{3}{17}$, $\frac{66}{201}$. (6) $2\frac{3}{4}$, $8\frac{10}{21}$, $\frac{19}{42}$, $90\frac{383}{504}$.
(7) $9\frac{10867}{16000}$, $3\frac{2}{3}$, $1\frac{49}{72}$. (8) $8\frac{26}{35}$, $2\frac{137}{144}$, $5\frac{13}{20}$.
(9) $\frac{83}{144}$, $\frac{377}{504}$. (10) $2\frac{5}{9}$.
(11) $3\frac{289}{720}$. (12) $11\frac{5}{18}$.

SET LI.

(1) $\frac{313}{1400}$, $\frac{313}{1050}$, $\frac{313}{840}$, $\frac{313}{700}$, $\frac{313}{600}$. (2) $2\frac{145}{198}$, $3\frac{1}{180}$, $3\frac{46}{165}$, $4\frac{13}{135}$.
(3) $\frac{9}{14}$, $\frac{1}{4}$, $\frac{4}{105}$. (4) $\frac{4}{45}$, $\frac{9}{13}$, $\frac{2}{3}$. (5) $1\frac{1}{3}$, $10\frac{2}{15}$.

(6) $27,\ 2\frac{17}{24}$. (7) $\frac{1}{11},\ \frac{9}{17}$. (8) $\frac{27}{50},\ \frac{1}{6}$.
(9) $78\frac{2}{3},\ 3$. (10) $360,\ 22$. (11) $1\frac{1}{2},\ 38$.
(12) $31\frac{1}{6}$. (13) $33\frac{489}{5000}$. (14) $7\frac{21}{50}$.
(15) $\frac{1859}{5400}$. (16) $\frac{17689}{90000000}$.

Set LII.

(1) $\frac{6}{41},\ \frac{30}{287},\ \frac{10}{287},\ \frac{6}{328}$. (2) $\frac{300}{5367},\ \frac{6}{487},\ \frac{76}{7792},\ \frac{16}{3409}$.
(3) $20,\ 17\frac{1}{7},\ 9\frac{3}{4}$. (4) $\frac{13}{32},\ 1\frac{7}{25},\ 1\frac{1}{15}$.
(5) $2\frac{1}{10},\ 1\frac{1}{8},\ 1\frac{13}{20}$. (6) $\frac{232}{425},\ \frac{8}{9},\ 1\frac{98}{1271}$.
(7) $2\frac{1}{10},\ 1\frac{11}{24},\ 9$. (8) $3\frac{3}{4},\ \frac{25}{38},\ \frac{608}{663}$.
(9) $\frac{4}{49},\ \frac{3}{28},\ 19\frac{1}{2},\ 1\frac{19}{74}$. (10) $\frac{196}{225},\ \frac{17}{64},\ \frac{19}{104},\ \frac{4}{9}$.
(11) $1\frac{7}{11},\ 1\frac{31}{45}$. (12) $20\frac{6}{11}$. (13) $1\frac{73}{194},\ 1\frac{91}{107}$.
(14) $1\frac{1}{5},\ 4\frac{1}{16}$. (15) $1\frac{559}{651}$. (16) $1\frac{79}{169}$.
(17) $71\frac{5}{29}$. (18) $\frac{94}{273}$. (19) $\frac{13}{20},\ \frac{686}{1159}$.
(20) $\frac{76}{105},\ \frac{2416}{3971}$.

Set LIII.

(1) $\frac{3}{4},\ \frac{5}{6},\ \frac{4}{5}$. (2) $\frac{6}{7},\ \frac{7}{8}$. (3) $\frac{3}{7}$.
(4) $\frac{4}{9},\ \frac{6}{11}$. (5) $\frac{7}{13},\ \frac{3}{5},\ \frac{9}{16},\ \frac{4}{7}$. (6) $\frac{2}{3}$.

Set LIV.

(1) $$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{4}}}},\qquad 1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{4}}}.$$

(2) $$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{2}}}}},\qquad 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{2}}}}.$$

(3) $$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}}}}},\qquad \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2}}}}}}}.$$

(4) $\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3}}}}}$, $\cfrac{1}{2+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3}}}}}}}$.

(5) $\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5}}}}}$, $1+\cfrac{1}{10+\cfrac{1}{1+\cfrac{1}{10+\cfrac{1}{1+\cfrac{1}{10}}}}}$.

(6) $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{32+\cfrac{1}{1+\cfrac{1}{2}}}}}$, $\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{3}}}}}}}}}$.

(7) $\frac{1}{4}$, $\frac{5}{21}$, $[\frac{31}{130}]$.

(8) 2, $2\frac{1}{2}$, $2\frac{4}{9}$, $2\frac{9}{20}$, $[2\frac{40}{89}]$.

(9) 3, $3\frac{1}{7}$, $3\frac{16}{106}$, $3\frac{16}{113}$ $[3\frac{421}{3044}]$.

(10) $\frac{1}{2}$, $\frac{12}{25}$, $\frac{13}{27}$, $\frac{25}{52}$.

(11) 1, $\frac{2}{3}$, $\frac{3}{4}$. $\frac{5}{7}$, $\frac{8}{11}$.

(12) 1, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{8}$, $\frac{7}{11}$.

(13) $\frac{1}{2}$, $\frac{2}{5}$, $\frac{5}{12}$, $\frac{12}{29}$, $\frac{29}{70}$, $\frac{70}{169}$.

SET LV.

(1) $5\frac{5}{11}$s. ; $\frac{1}{16}$s.

(2) £$\frac{7}{180}$; £$\frac{1}{420}$; £$\frac{7}{15}$.

(3) 5s. $2\frac{1}{2}$d. ; $5\frac{1}{4}$d.

(4) $\frac{1}{640}$ da. ; 135 sec. ; $\frac{1}{4480}$ wk.

(5) 90 lb. ; $\frac{9}{224}$ ton.

(6) $\frac{576}{1925}$ lb. avoird.

(7) 140 yd. ; $\frac{7}{88}$ mi.

(8) $\frac{5}{1536}$ ac. ; $20418\frac{3}{4}$ sq. in.

(9) $4\frac{1}{2}$ pt. ; $\frac{9}{128}$ bus.

(10) $2\frac{41}{192}$ oz. troy.

(11) 12 hr. 26 min. 40 sec.

(12) 12 cwt. 1 qr. 4 lb.

(13) 1 ft. 2 in.

(14) 6 sq. po. 14 sq. yd. 72 sq. in.

(15) 1 pk. 1 gall. 2 qt.

(16) 2 mi. 17 po. 4 yd. 10 in.

(17) 8 cub. yd. 6 cub. ft. 1296 cub. in.

(18) 3 ac. 29 sq. po. 2 sq. yd. 6 sq. ft. 108 sq. in.

(19) 6 lb. 5 oz. $13\frac{1}{11}$ dr.

(20) 3 po. 1 ft. $7\frac{4}{5}$ in.

SET LVI.

(1) £$\frac{1}{8}$; £$\frac{247}{480}$; £$7\frac{5}{8}$; £$2\frac{1}{480}$.
(2) $\frac{6}{13}$s. ; $16\frac{17}{24}$s. ; $132\frac{6}{13}$s.
(3) $1\frac{23}{28}$ cwt. ; $3\frac{15}{16}$ cwt. ; $61\frac{1}{4}$ cwt.
(4) $\frac{77}{180}$ da. ; $\frac{3}{7}$ da.
(5) $\frac{9}{168}$ fur. ; $\frac{7}{44}$ fur.
(6) $26\frac{1}{99}$ links.
(7) $1\frac{7}{8}$ ac. ; $\frac{17}{64}$ ac.
(8) $\frac{15}{16}$ gall. ; $5\frac{3}{4}$ gall.
(9) $2\frac{1}{12}$ lb. troy.
(10) $1\frac{743}{1225}$ lb. avoird.

SET LVII.

(1) $\frac{61}{800}$ of £100 ; $\frac{7}{9600}$ of £100.
(2) $\frac{103}{840}$.
(3) $\frac{19}{20}$.
(4) $\frac{8}{11}$ of 6 ac. 3 ro. 9 sq. po.
(5) $\frac{2}{7}$ of 7 cwt. $3\frac{1}{2}$ st. ; $\frac{9}{11}$ of 6 cwt. 15 lb. 8 oz.
(6) $\frac{1331}{7840}$.
(7) $\frac{9024}{10325}$.

SET LVIII.

(1) $\frac{7}{45}$. (2) $\frac{1}{42}$. (3) $\frac{1}{91}$. (4) $10\frac{81}{168}$.
(5) $102\frac{1}{2}$ carats. (6) $1\frac{13}{35}$. (7) 10. (8) $1\frac{42}{85}$.
(9) $2\frac{1}{4}$d. (10) £17 2s. $2\frac{2}{3}$d. (11) £4 8s. 8d. (12) £2 13s. 4d.
(13) $5\frac{2}{9}$. (14) £10 0s. $11\frac{1}{2}$d. (15) £3080.
(16) £1443 12s. (17) 30 da. (18) $7\frac{23}{91}$. (19) $1\frac{3}{7}$ hr.
(20) £89512 10s. (21) 5. (22) $\frac{469}{255000}$. (23) $\frac{80}{1891}$.
(24) $\frac{2}{25}$. (25) $1\frac{23}{160}$ lb. (26) £1680. (27) 88.
(28) 6 hr. (29) $4\frac{83}{1794}$. (30) $\frac{441}{1161}$ gall. or $3\frac{45}{1161}$ pt.
(31) £28. (32) $\frac{1675}{13122}$. (33) £4950. (34) £20 12s. 6d.
(35) $\frac{933}{3040}$. (36) 14 lb. (37) $\frac{1605}{3311}$. (38) 1.
(39) £29900. (40) $\frac{18}{55}$.

SET LIX.

(1) Seven thousandths. Three hundredths. Four ten-thousandths. One millionth. Five hundred-thousandths.

(2) Nine ten-millionths. Seven thousand-millionths. Three billionths.

(3) One whole unit and five hundredths. Ten whole units and a tenth. Two hundred whole units and two hundredths. Three thousand whole units and three ten-thousandths.

(4) Thirty-one hundredths. Three hundred and twelve thousandths. Three hundred and twelve thousandths and three ten-thousandths. Thirty-one thousandths and twenty-three hundred-thousandths. Three thousandths and thirteen millionths.

(5) Three hundred and twelve millionths. Thirteen ten-thousandths. Two hundred and thirty-one thousandths two hundred and thirty-one millionths. Six hundred and eighteen thousandths one hundred and eighty-six millionths.

(6) One hundred and thirty thousandths nineteen millionths. Three

hundred thousandths three millionths. Twelve thousandths twelve millionths twelve thousand-millionths.

(7) **Fifty thousandths six hundred and seven ten-millionths. Four thousand and one ten-millionths. One hundred thousandths five hundred-millionths.**

(8) **Ten million whole units and seven ten-millionths. Eleven millionths one ten-thousand-millionth.**

(9) **Six millionths three hundred and four thousand and twenty-seven billionths. Seven thousandths three hundred millionths one thousand and two billionths.**

(10) **Ten millionths one thousand billionths one ten-billionth. One hundred thousand billionths one hundred-billionth.**

Set LX.

(1) .3 ; .004 ; .000009 ; .00007.
(2) .00000001 ; .000000000005 ; .000000000000007.
(3) 4.7 ; 16.03 ; 9000.009 ; 100.0000001.
(4) .015 ; .00027 ; .000402 ; .000000001017.
(5) .203014 ; .05000007 ; .000606002012.
(6) .00001 ; .003003 ; .0407 ; 20.16.
(7) .015202 ; .110012 ; .000015000202.
(8) 20.5 ; .017011 ; 10.0000040003.
(9) .0300416 ; .760201 ; .0504018.
(10) 41.026 ; .001043 ; .3000265.

Set LXI.

(1) 5, 2, 4, 625, 8, 3125, 32.
(2) $\frac{5}{10}$, .5 ; $\frac{6}{10}$, .6 ; $\frac{68}{100}$, .68.
(3) $\frac{375}{1000}$, .375 ; $\frac{425}{1000}$, .425 ; $\frac{6875}{10000}$, .6875.
(4) $\frac{9375}{100000}$, .09375 ; $\frac{88}{1000}$, .088 ; $\frac{25625}{1000000}$, .025625.
(5) $\frac{192}{10000}$, .0192 ; $\frac{96}{1000000}$, .000096 ; $\frac{984375}{1000000}$, .984375.
(6) 3, 6, 7, 9, 11, 12, 13, 14, 15, 17, 18, 19.

Set LXII.

(1) .8125, .73125, .992.
(2) .04, .0875, .016.
(3) .0296875, .0009375, .00544.
(4) .007890625, .0000064, .000107421875.

Set LXIII.

(1) .428, .363, .444.
(2) .692, .148, .62.
(3) .034782, .019801, .107843.
(4) .970873, .003996, .000533.
(5) $.\dot{2}$, $.\dot{5}$, $.\dot{8}$.
(6) $.\dot{6}$, $.4\dot{6}$, $.41\dot{6}$, $.3\dot{7}$.

(7) $.\dot{3}\dot{1}$, $.\dot{1}\dot{4}$, $.\dot{0}\dot{2}$, $.\dot{2}\dot{7}$. (8) $.\dot{2}0\dot{6}$, $.\dot{0}3\dot{4}$, $.\dot{0}0\dot{5}$.
(9) $.\dot{0}01\dot{6}$, $.\dot{0}0201\dot{4}$. (10) $.\dot{4}5\dot{9}$, $.\dot{0}49\dot{5}$, $.\dot{0}4187\dot{7}$.
(11) .090909, .076923, .058823, .052631.
(12) 3.142...., 3.14150...., 3.14159.... Beginning with the lowest number the order of magnitude therefore is $\frac{333}{106}$, $\frac{355}{113}$, $\frac{22}{7}$.
(13) $.0\dot{6}$, $.\dot{1}4\dot{2}$, $.31842\dot{7}$.

Set LXIV.

(1) $\frac{7}{10}$, $\frac{3}{100}$, $\frac{1}{2000}$, $\frac{1}{125000}$. (2) $\frac{3}{4}$, $\frac{23}{200}$, $\frac{47}{2000}$, $\frac{83}{40000}$.
(3) $\frac{1}{200}$, $\frac{1}{400}$, $\frac{2}{3125}$, $2\frac{1}{2}$. (4) $2\frac{1}{4}$, $7\frac{1}{8}$, $3\frac{3}{8}$, $1\frac{5}{8}$.
(5) $1\frac{3}{250}$, $14\frac{3}{200}$, $101\frac{3}{16}$, $400\frac{61}{1600}$.

Set LXV.

(1) $\frac{1}{9999}$, $\frac{1}{99999999}$, $\frac{1}{999999}$. (2) $\frac{7}{9}$, $\frac{1}{3}$, $\frac{7}{99}$, $\frac{7}{11}$, 1, $\frac{1}{11}$.
(3) $\frac{68}{111}$, $\frac{7}{333}$, $\frac{1}{333}$, $\frac{1101}{1111}$. (4) $4\frac{1}{27}$, $\frac{11}{909}$, $\frac{707}{33333}$, $\frac{1001001}{1111111}$.

Set LXVI.

(1) $\frac{5}{6}$, $\frac{49}{60}$, $\frac{26}{33}$, $\frac{233}{720}$. (2) $\frac{311}{9900}$, $\frac{667}{333000}$, $\frac{1}{220}$.
(3) $2\frac{1}{1665}$, $2\frac{1}{1650}$, $2\frac{1}{1800}$. (4) $\frac{69}{99}$, $\frac{7001}{9990}$, $\frac{773}{9900}$.
(5) $\frac{8}{9}$. (6) $\frac{7}{99}$. (7) 1.
(8) $\frac{1}{100}$. (9) $\frac{3}{1000}$. (10) $5\frac{3}{8}$.

Set LXVII.

(1) 7042.341763. (2) 2145.001. (3) 9.663.
(4) 1. (5) 21.89833. (6) 9.4895.
(7) .0161239. (8) .101800149001. (9) 9.40178....
(10) 6.691577.... (11) 2.5003434343....
(12) $6.235\dot{2}0911\dot{8}$. (13) $2.275\dot{1}7535897733\dot{9}$.
(14) $2.0009\dot{4}\dot{6}$. (15) 1017.54641.

Set LXVIII.

(1) 17.3188. (2) .022849. (3) .00136.
(4) .000813. (5) .09912. (6) .000006.
(7) 2968.35. (8) 1.23469. (9) 11.6593873.
(10) 99.9999999. (11) .0133052. (12) $.029\dot{0}43995799$.
(13) 2.657575...., .611347....
(14) $.88\dot{1}19\dot{7}$, $1.22\dot{6}\dot{9}$. (15) $.59\dot{9}9344\dot{7}$, $1.6\dot{0}6552\dot{0}$.

Set LXIX.

(1) .7, 7, 70000, 70070. (2) .101, 10100, 1.
(3) 15426.72. (4) 2.7261312.

(5) 100. (6) 90. (7) 3000.
(8) .0010792. (9) .0000004. (10) .00012081208.
(11) 3.03303. (12) .102. (13) 912.0336.
(14) 912.0336. (15) 27000. (16) 27000.

Set LXX.

(1) 2.2814..., 4.5034... (2) .134..., .977...
(3) 7.015351..., 6377.457017... (4) 2.780...., $2.7800\dot{8}$.
(5) .001..., $.001\dot{3}$. (6) 4.824..., $4.8\dot{2}4\dot{3}$.
(7) .055..., $.05\dot{5}\dot{7}$.

Set LXXI.

(1) .001, .00001, .00000001. (2) 10.1, .101, .00101.
(3) 2.0212, .0000020212. (4) .4.
(5) .013. (6) .7872. (7) .0013802.
(8) .00087. (9) .00000003. (10) .390625.
(11) .184. (12) 200. (13) 18.4.
(14) 4001.2. (15) 437.5. (16) .0125.
(17) .0013. (18) .0015. (19) .000026.
(20) .00046. (21) $.0000\dot{1}9\dot{0}$. (22) $1.9814\dot{6}$.
(23) .017. (24) $5.\dot{7}6190\dot{4}$.
(25) $.56\dot{3}95348837209302325581\dot{1}$.

Set LXXII.

(1) £.0009075, £.013335. (2) .12084 cr., 2.4168 cr.
(3) 19.3536 min., .00192 wk. (4) .03362625 cwt., 59.25824 oz.
(5) .0000063125 mi., .39996 in.
(6) $85.5638\dot{9}6\dot{3}$ sq. ft., $.0019642\dot{7}6\dot{7}$ ac.
(7) 7s. 3d. (8) £2 15s. $6\frac{3}{4}$d.
(9) $9\frac{3}{4}$d. (10) £1 9s. $4\frac{3}{4}$d. $\frac{6898}{10000}$ far.
(11) 3 hr. 3 min. 9 sec. (12) 1 ton 3 cwt. 2 qr. 7 lb.
(13) 2 ft. $1\frac{718}{1000}$ in. (14) 3 ro. 27 sq. po.
(15) 1 gall. 3 qt. 1 pt.
(16) 1 cub. yd. 26 cub. ft. 270 cub. in.
(17) £1 16s. $2\frac{1}{2}$d. (18) 3 cwt. 1 qr. 6 lb.
(19) 14 sq. in. (20) 1 ac. 2 ro. 26 sq. po. 11 sq. yd.

Set LXXIII.

(1) £1.8 ; £.6375 ; £1.3125.
(2) £12.871875 ; £2.009375 ; £$6.07708\dot{3}$.
(3) .215625 ton ; .27734375 lb.

(4) 40.1875 cwt. ; .373046875 st.
(5) .73125 da. ; $360.28\dot{3}$ min.
(6) $4.58\dot{3}$ yd. ; $1.367329\dot{5}\dot{4}$ mi.
(7) $8.91\dot{6}$ fathoms ; $.1571\dot{9}\dot{6}$ mi.
(8) .60625 ac. ; $.2\dot{5}\dot{4}$ ac. ; $.54\dot{3}\dot{9}$ ac.
(9) $.441550\dot{9}2\dot{5}$ cub. yd. ; .921875 qr.
(10) .46953125 cwt. ; $33.791\dot{6}$ yd.

SET LXXIV.

(1) .70875 of £10 ; .1484375 of £10.	(2) $.0152\dot{7}$ of £10 10s.
(3) $.7\dot{3}8095\dot{2}$.	(4) .6875 of 12 st.
(5) $.\dot{1}\dot{0}$ of $2\frac{1}{8}$ da.	(6) $4333.\dot{8}4615\dot{3}$ fr.

SET LXXV.

(1) 3000 m.; 6500 m.; 2343 m.; 802 m.; 5055 m.
(2) £.61; £.053; £3.208.
(3) 50 fl.; .5 fl.; .05 fl.; 132.03 fl.

(4) £116.914.	(5) 21109 m.	(6) £10.474.	
(7) 495 fl.	(8) £93 6 fl. 3 c. 6 m.	(9) 2 fl. 2 c. 4 m.	
(10) £2 6 fl. 1 c.	(11) £30 8 fl. 3 c. $3\frac{1}{3}$ m.	(12) $3\frac{3}{4}$ lb.	(13) 74.

SET LXXVI.

(1) £.6.	(2) £.65.	(3) £.675.	(4) £.875.
(5) £.325.	(6) £1.4.	(7) £3.375.	(8) £2.028...
(9) £1.05.	(10) £1.810...	(11) £2.667...	(12) £5.271...
(13) £4.185...	(14) £1.392...	(15) £2.587...	(16) £10.529...
(17) £7.878...	(18) £11.597...	(19) £10.076...	(20) £12.051...
(21) £20.027...			

SET LXXVII.

(1) 13s.	(2) 17s.	(3) 2s. 6d.	(4) 6s. 6d.
(5) 13s. 6d.	(6) 19s. 6d.	(7) £8 0s. 6d.	(8) £3 1s.
(9) £6 5s. 6d.	(10) £3 3s. 6d.	(11) £6 7s. 9d. nearly.	
(12) £4 8s. $3\frac{3}{4}$d. nearly.		(13) 12s. $8\frac{1}{4}$d. nearly.	
(14) £1 1s. $6\frac{3}{4}$d. nearly.		(15) £2 14s. $1\frac{3}{4}$d. nearly.	
(16) £5 15s.	(17) £217 15s.	(18) 4s. $5\frac{1}{4}$d.	
(19) 12s. $7\frac{1}{2}$d.	(20) 7s. 3d.	(21) 19s. $3\frac{3}{4}$d.	

SET LXXVIII.

(1) 3.6 kilom.; 36000 decim.; 3600000 millim.
(2) .0156 kilom.; 15600 millim.
(3) 11 decim. 34 millim.

(4) 46 decim. 2 centim.
(5) 31.056 kilom.; 31056000 millim.
(6) 1000000 times; 1000 times.
(7) 170160 decim.; 1701.6 decam.
(8) .403 metre; 403 millim.
(9) 6.4 metres, nearly; 3.2 metres, nearly; 25.4 centim., nearly.
(10) 26.553.... kilom.; 6.839..... kilom.; 8.138 kilom., nearly.
(11) 11 yd. 1 ft. $5\frac{3}{8}$ in., nearly; 45 mi. $1728\frac{1}{2}$ yd., nearly.
(12) 1000 kilom. (13) 7500 kilom.

SET LXXIX.

(1) 114.3 ares; 11430 centiares.
(2) .1411 are; 14.11 sq. metres; 1411 sq. decim.
(3) 27300 sq. metres; 14600 sq. metres; 170 sq. metres.
(4) 10000; 1000000.
(5) 3.762... sq. metres; 9.29 sq. metres, nearly; 1.2334.... sq. metre.
(6) 24.28.... ares; 6.7782 hectares; 1.052... hectare.
(7) 23.25.... sq. in.; 13 ac. 3 ro. 22.05... sq. po.

SET LXXX.

(1) 117640 cub. centim.; .11764 cub. metres.
(2) 17 cub. decim. 430 cub. centim.; 31 cub. decim. 426 cub. centim.
(3) 1000000 times.
(4) 3104 centil.; 31 litres 4 centil.
(5) .03142 hectol.; 3142 millil.
(6) 3000; 300.
(7) 1530 cub. centim.; 416.3 cub. metres.
(8) 2.35959... cub. decim.; 15.65825... cub. metres; 47.7062.... litres.
(9) 21.445... hectol.; 3.7256.... hectol.; 3.975.... litres.
(10) 16.7824... cub. in.; 13.08... cub. yd.
(11) 3 gall. 1 qt. 1.6... gill; 2 qr. 0.507... bus.

SET LXXXI.

(1) 51700 grams; .517 quintal; 51700000 millig.
(2) 4 grams 10 millig.; 31 grams 40 millig.
(3) 17009 kilog.; 4 kilog.; 10 kilog.
(4) 10000000; 350000.
(5) 14.3 grams; 30 millig.; 16 kilog.
(6) 3.16 millil.; 400 litres; 14 centil.; 10 hectol.
(7) 3.4019... kilog.; 63.503.... kilog.; 2.3813... kilog.
(8) 34.473... kilog.; 1.77808... metric ton; 2.4131... quintals.
(9) 69.853... kilog.; 21.0575... metric tons; .37324... kilog.
(10) 15 lb. 6.9... oz.; 19.684... tons; 3 cwt. 16.739... lb.

SET LXXXII.

(1) 50.9996 quintals. (2) 25 millil. (3) 2.2 hectol.

(4) 140000 times. (5) 50. (6) £16.

(7) £1.25. (8) £660. (9) 100.

(10) 4 decim. (11) 60.512 kilog. (12) 160 cub. decim.

SET LXXXIII.

(1) $.05\dot{3}57142\dot{8}$.

(2) A deficit of £1.46 is not accounted for.

(3) 23 hr. 15 min. 18.6... sec.; 3 da. 2 hr. 28 min. 57.2... sec.

(4) 5 hr. 48 min. 49.65... sec. (5) £864 15s. $0\frac{2}{25}$d.

(6) $.41\dot{6}$. (7) 15130.340... (8) .2597.

(9) 1 farthing. (10) .009009. (11) £175 16s. 9.8...d.

(12) 1s. 1.65...d. (13) .00000025. (14) 732.5... millim.

(15) 11 yr. 317 da. 14 hr. 2 min. 24 sec.; 29 yr. 174 da. 5 hr. 16 min. 48 sec.

(16) 1946.872. (17) .001968d. (18) $4712\frac{1}{2}$ kilog.

(19) 1.14 ton. (20) 77. (21) 17.329.... cub. in.

(22) 18.075 kilom. (23) £73.175. (24) $2.5\dot{3}9866148777\dot{0}$.

(25) 6d. (26) £58.21. (27) 96059601.

(28) 1.018845488. (29) 5.363... yd. (30) 25.125 fr.

(31) £250.782. (32) 15.5711875, $15\frac{9139}{16000}$.

(33) 44 kilom. or (44 − 1.375) kilom. (34) £166 13s. 4d.

(35) $9.48\dot{2}$. (36) $105.\dot{3}$, $105\frac{1}{3}$.

(37) 216000 oz. (38) .12499... (39) 206.73792 metres.

(40) 20.795... fr. (41) £6987.9457....

(42) No gain, but a loss (1) of £6.5; (2) of £14.069....

(43) 2.7182818.... (44) 230.4675 fr. (45) 7200. (46) 300.

(47) 14294 lb. (48) .6931...; 1.1775.... (49) $\frac{888}{999}$.

(50) The numbers of the last column of the table are in order:— 2.4...; 3.6...; 31.2...; 3.8...; 234.3...; 604.6...; 6.8...; 722.2...; 222.7...

SET LXXXIV.

(1) 4s. 9d. (2) 5s. 3d. (3) 6s. 6d. (4) 1s. 3d.

(5) 12s. 6d. (6) 17s. 6d. (7) £1 5s. 3d. (8) £2 0s. 9d.

(9) £3 19s. 3d. (10) £5 16s. 6d. (11) £7 5s. 9d. (12) £9 4s. 6d.

(13) 15s. (14) 17s. 6d. (15) 18s. 9d. (16) £1 8s. 6d.

(17) £2 9s. (18) £1 12s. 6d. (19) £1 7s. 6d. (20) £5 17s.

(21) 7s. 6d. (22) 17s. (23) £1 1s. (24) £2 17s. 9d.

(25) 11s. 3d. (26) £1 3s. $11\frac{1}{2}$d. (27) 13s. $7\frac{1}{2}$d. (28) 5s. $4\frac{3}{4}$d.

(29) £1 11s. 3d. (30) £6 9s. 2d. (31) £2 11s. (32) £9 9s.

Set LXXXV.

(1) 1s. $3\frac{3}{4}$d. (2) 6s. 1d. (3) 5s. $5\frac{1}{4}$d.
(4) 9s. $0\frac{1}{2}$d. (5) 9s. 11d. (6) 3s. $6\frac{1}{4}$d.
(7) 5s. $11\frac{1}{2}$d. (8) £1 16s. $1\frac{1}{2}$d. (9) £80 19s. 9d.
(10) £164 6s. $8\frac{1}{2}$d. (11) £1 9s. $9\frac{3}{4}$d. (12) £495 13s. 9d.
(13) £7 8s. $3\frac{3}{4}$d. (14) £33 7s. $5\frac{1}{4}$d. (15) 6s. 9d.
(16) £2 3s. 10d. (17) 1s. 1d. (18) £1 6s. 4d. nearly.
(19) £6 11s. nearly. (20) £2 17s. $7\frac{3}{4}$d. nearly.

Set LXXXVI.

(1) £14 16s. $0\frac{1}{2}$d. (2) £236 7s. 6d. (3) £15 5s. $8\frac{7}{16}$d.
(4) £60 2s. $3\frac{1}{2}$d. (5) 8s. $5\frac{1}{4}$d. (6) £85 15s. $1\frac{1}{4}$d.
(7) £41 12s. $3\frac{1}{2}$d. (8) £48 15s. $8\frac{11}{20}$d. (9) £9 19s. $5\frac{7}{20}$d.
(10) £1 12s. $2\frac{13}{20}$d. (11) £134 9s. 2d. (12) £1 7s. $0\frac{2}{15}$d.
(13) £6 17s. $1\frac{7}{8}$d. (14) £158 17s. $3\frac{3}{5}$d. (15) 15s. $11\frac{2}{3}$d.
(16) £12 12s. $5\frac{11}{54}$d. (17) 7s. $9\frac{23}{27}$d. (18) £5 19s. $10\frac{1}{8}$d.
(19) £2 15s. $9\frac{3}{4}$d. (20) £26 11s. $5\frac{71}{160}$d. (21) £8 11s. $1\frac{199}{224}$d.
(22) £69 6s. $8\frac{1}{2}$d. (23) £36 7s. $11\frac{11}{32}$d. (24) £56 1s. $9\frac{13}{20}$d.
(25) £8 11s. $11\frac{173}{560}$d. (26) £3 18s. $4\frac{83}{112}$d. (27) £184 9s. $6\frac{1}{5}$d.
(28) £33 6s. $3\frac{1161}{1780}$d. (29) £29 12s. $7\frac{17}{220}$d. (30) £29 0s. $10\frac{24}{25}$d.
(31) £23 10s. $6\frac{6}{11}$d. (32) £1 19s. $5\frac{1}{8}$d.

Set LXXXVII.

The full amounts of the bills are :—

(1) £4 16s. $4\frac{1}{2}$d. (2) £33 14s. 8d. (3) £142 6s. 9d.
(4) £5 18s. 9d. (5) £56 19s. 9d. (6) £1 10s. 8d.
(7) 19s. $11\frac{3}{10}$d. (8) £47 2s. 10d. (9) £469 3s. $1\frac{1}{4}$d.
(10) £684 17s. (11) £7 4s. $8\frac{3}{4}$d. (12) £1 9s. $10\frac{3}{4}$d.
(13) £321 17s. $0\frac{3}{4}$d. (14) £1 5s. $10\frac{15}{16}$d. (15) £319 0s. $6\frac{3}{8}$d.
(16) £672 4s. $10\frac{67}{80}$d. (17) £116 5s. $0\frac{17}{40}$d. (18) £53 15s. $3\frac{3}{4}$d.

Set LXXXVIII.

(1) 14s. 5d. (2) 7s. 9d. (3) 14s. 7d.
(4) 16s. 8d. (5) 12s. (6) £2 8s. 6d.
(7) £3 15s. (8) £2 19s. $2\frac{4}{13}$d. (9) £4 12s.
(10) 5s. $4\frac{3}{4}$d. (11) 3s. 9d. (12) 15s. 5d.
(13) 4s. 4d. (14) £4 15s. 6d. (15) 1s. $6\frac{2}{81}$d.
(16) £4 6s. 8d. (17) The second kind is the dearer by 2d. per yd.
(18) £1670 18s. (19) 14s. $0\frac{344}{1367}$d. (20) 2s. $8\frac{22784}{31103}$d.

Set LXXXIX.

(1) 17s. $5\frac{13}{23}$d. (2) £359 4s. (3) £1 4s. 8d.

(4) 8.925 fr. (5) £155 11s. $0\frac{28}{141}$d. (6) £4 12s. 1d.
(7) £34 3s. $3\frac{9}{49}$d. (8) $4\frac{1}{2}$d. (9) £3 17s. $9\frac{1}{3}$d.
(10) 18s. $2\frac{5}{8}$d. (11) £53 6s. 8d. (12) £5 13s. $1\frac{1}{4}$d.
(13) 1387.5 fr. (14) £262 19s. $0\frac{1}{2}$d. (15) £1 7s. $11\frac{1}{2}$d.
(16) £2 13s. 4d. (17) £19 5s. (18) £15 5s. $2\frac{1}{4}$d.
(19) £23 3s. 6.28...d. (20) £2 7s.

Set XC.

(1) $38\frac{43}{84}$ mi. (2) $52\frac{47}{100}$ tons. (3) $1772\frac{5}{8}$.
(4) 7449.5625 gall. (5) 7.965... decal. (6) 32.
(7) $99\frac{3}{28}$ cub. ft. (8) 18 hr. (9) 14 yd.
(10) $50\frac{2}{3}$ bus. (11) $68\frac{7}{11}$ mi. (12) $39\frac{1}{5}$ lb.
(13) 133° 41′ 24″ nearly. (14) 9.83... litres. (15) $38\frac{8}{9}$ metres.
(16) .1089... rad. (17) $19\frac{9}{13}$ da. (18) $2\frac{73}{8080}$ hr.
(19) 5.75 florins; 36.036 fr. (20) £5 18s. 8d.
(21) 19s. $7\frac{1}{2}$d. nearly. (22) $\frac{26}{35}$ lb. (23) $10\frac{5}{8}$ da.
(24) $2\frac{28}{29}$ lb. (25) £3 16s. 10d. (26) 28.
(27) 7.576... oz. (28) 2 mi. $666\frac{2}{3}$ yd. (29) 4.3508... yd.
(30) 218.5505 tons. (31) 157.623... lb. (32) 16 ft. $1\frac{7}{81}$ in.
(33) $1\frac{1}{384}$ lb.
(34) 998.864 lb., 277.752 lb., 27.384 lb.; 1305.48... lb.
(35) 15 ac. 32 sq. po.
(36) $1\frac{61338617}{61479883}$ sec. (*i.e.* very nearly 2 sec.) past 1 P.M.
(37) $3\frac{1}{3}$d. (38) 29.37...; 4.38... (39) 42.08; 17.88...

(40)

	Réaumur.	Fahrenheit.	Centigrade.
1	15.00	65.75	18.75
2	16.16	68.36	20.20
3	17.96...	72.42	22.45
4	19.93...	76.85	24.91...
5	33.50	107.37...	41.87...
6	42.65	127.96...	53.31...

Set XCI.

(1) £1047 1s. $10\frac{1}{2}$d. (2) 11s. 9d. (3) 2s. $9\frac{1}{2}$d.
(4) £149 16s. 3d. (5) 2d. (6) 12 cub. ft. $92\frac{4}{7}$ cub. in.
(7) $9112\frac{1}{2}$ cub. ft. (8) 2s. (9) £91 18s. $0\frac{3}{10}$d.
(10) £3421 5s. (11) $2\frac{2}{5}$ cub. ft; 1s. $8\frac{2}{5}$d.
(12) £8 16s. $5\frac{13}{35}$d. If the sum for maintenance were £10001 5s., as was intended, the result would be 5s.
(13) $4\frac{1}{2}$ gall.; 1675350 gall.

SET XCII.

(1) £81 7s. 6d. (2) £89 1s. 3d. (3) £11 14s.
(4) £6 11s. 3d. (5) 16 ac. 1 ro. (6) £11 2s. $5\frac{5}{8}$d.
(7) £37 16s. (8) £214 7s. 6d. (9) £125 18s. $3\frac{21}{50}$d.
(10) £175 10s. (11) 2925 lb. (12) 12288 gall.
(13) 16 da. (14) 33 hr. (15) 22 da.
(16) 2 hr. (17) 21 da. (18) 10.
(19) 84. (20) 119 mi. (21) 11 hr.
(22) 80; 3d. (23) £892 10s. (24) £1 17s. 6d.
(25) $16\frac{14}{25}$ hr. (26) 3 qt. $1\frac{14887}{348859}$ pt. (27) 3 ft. $10\frac{2478}{356825}$ in.
(28) £487 4s. (29) $114\frac{27}{32}$ hr.
(30) $5\frac{2}{3}$; that is, more than five men would be required, and six could do more than the specified work in the given time.
(31) 2 cwt. 2 qr. 3 lb. 15.056 oz. (32) 10. (33) 29.9...

SET XCIII.

(1) 31415.9 sq. ft. (2) £5 17s. 4d. (3) 1610 ft.
(4) 32 times. (5) $549\frac{88}{113}$ sq. ft. (6) £4 10s. $3\frac{3}{8}$d.
(7) 99.225 metres. (8) 257.6 ft.
(9) They are inversely proportional.
(10) *Time* taken to do a work, and *number of men* employed.
(11) The second is inversely proportional to the third power of the first.
(12) 28.2743.... in. (13) 210.9375 sq. in. (14) 5 min. $27\frac{21}{317}$ sec.
(15) 1s. $1\frac{7}{32}$d. (16) £4 6s. 0.192d. (17) £28 16s.

SET XCIV.

(1) £2 2s. 9d. (2) £21 3s. 9d. (3) £16 0s. 6d.
(4) £207 14s. 1d. (5) £7259 1s. 3d. (6) Total=£29 13s. $6\frac{3}{8}$d.
(7) £203 11s. $0\frac{13}{20}$d. (8) £202 19s. 9d. (9) £150 15s.
(10) £7910 2s. (11) 4 per cent. (12) £36 9s. $7\frac{4}{11}$d.
(13) $4\frac{1}{2}$ per cent. (14) £579 2s. 6d. (15) $\frac{1}{4}$ per cent.
(16) £313 11s. $10\frac{1}{2}$d. (17) £205 4s. 2d. (18) 15 per cent.
(19) £688 4s. $8\frac{8}{17}$d. (20) £1200. (21) £1560.
(22) £68 17s. (23) £4016 13s. 4d. (24) £100000.

SET XCV.

(1) 65; 10; $8\frac{1}{3}$; 6. (2) 40.
(3) $92\frac{1}{2}$. (4) 96; 60.

(5) The items of the last column are in order:—91.8...; 87.0...; 89.7...; 92; 92.3...; 100; 90.3...

(6) Brown's, £4594 10s.; Robinson's, £918 18s.; the other's, £612 12s.

(7) 31.3... ; 1.9...; 55.4... ; 2.7... ; 3.0... ; 5.4...
(8) 35. (9) £210 18s. 9d.
(10) Nitre, 76; charcoal, 14; sulphur, 10.
(11) Copper, 81; tin, 19. (12) 6.5025 gall.
(13) 40; 4. (14) 3.6; .22.

(15)

			41.25 tons 38.465 ,, 293.36 ,,	2.0625 tons 1.92325 ,, 14.668 ,,
Total	1349 tons	27.65...	373.075 tons	18.65375 tons

(16) 81.8... (17) 13.5... (18) 31.8...
(19) 52.4... (20) 73.274. (21) 62.85.
(22) First, 60; second, 30; third, 10. (23) 150 ac.
(24) A, $33\frac{1}{3}$; B, $22\frac{2}{9}$; C, $44\frac{4}{9}$. (25) 260.

Set XCVI.

(1) 50. (2) 100. (3) 50. (4) 40.
(5) 5. (6) $13\frac{23}{29}$. (7) $3\frac{17}{31}$.
(8) England and Wales, increase, 13.17... ; Scotland, increase, 9.71... ; Ireland, decrease, 6.72... (9) 4.
(10) 5s. $5\frac{1}{4}$d. (11) Gain, 35 per cent. (12) £121 10s.
(13) 16. (14) 34419014[.6...]. (15) £1857 2s. $10\frac{2}{7}$d.
(16)

District.	Population in 1871.	Population in 1861.	Increase.		Decrease.	
			Actual.	Percentage.	Actual.	Percentage.
B			15143	33.29...		
C			1552	4.23...		
D					3931	13.11...
E			4746	16.53...		
Total	140782	158292	17510	12.43...		

(17) 70098[.336]. (18) £30 10s. (19) 3s. 4d.
(20) $6\frac{1}{2}$d. (21) 18124[.2...]. (22) 17816[.9832].
(23) £1 5s. 8d. (24) £68 19s. $11\frac{3}{8}$d. (25) £484.
(26) £3 17s. 11d. (27) Increase, .87... (28) 3s. $3\frac{3}{4}$d.
(29) 8. (30) 3s. 6d. (31) Loss, 1 per cent.
(32) None. (33) $5\frac{1}{2}$d. (34) 4s. $1\frac{1}{2}$d.
(35) $4\frac{7}{20}$d. (36) 30.
(37) He gained $4\frac{8}{25}$ per cent. (38) 40.
(39) A decrease of $8\frac{1}{28}$ per cent. (40) 9.363....
(41) A loss of 2 per cent. (42) Increase of $2\frac{2}{49}$ per cent.
(43) .962... (44) $3\frac{3}{4}$ oz.

Set XCVII.

(1) 8. (2) 5 ft. $3\frac{1}{4}$ in. (3) £2 17s. 8d. (4) £20 15s. 8d.

(5) The items of the last column should be:—72.4...; 87.9...; 92.3...; 91.2...; 124.9...; 96.3...

(6) 1d. (7) $\frac{205}{2304}$. (8) $11\frac{249}{864}$ yr. (9) $6\frac{2}{3}$ mi. per hr.

(10) $.0\dot{2}61904\dot{7}$ in.

(11) The items of the last line should be:—59.3...; 43.0...; 30.134...; .005...

(12) $71\frac{7}{48}$. (13) 4.7; 3.9... (14) $1.5291\dot{6}$ fr.

(15) 21.25. (16) $10\frac{2}{3}$ carats.

(17) 3 lb. of coffee to 2 lb. of chickory.

(18) 10 bus. at 5s. 6d., and 20 bus. at 6s.

(19) 6 lb. at 2d. to 8 lb. at $2\frac{1}{2}$d. (20) 12.

(21) 40, 60; $9\frac{1}{3}$ lb.; $44\frac{4}{5}$ lb., $67\frac{1}{5}$ lb. (22) 14. (23) $42\frac{6}{7}$.

(24) 1 part at 2s. 6d., 3 parts at 2s. 8d., 2 parts at 3s.; or 2 parts at 2s. 6d., 3 parts at 2s. 8d., 3 parts at 3s.; or in various other ways.

(25) 4 parts of the first coffee, 27 parts of the second, and 4 paits of chickory.

(26) 60 parts of lead, 5 parts of antimony, 8 parts of copper; or 76 parts of lead, 9 parts of antimony, 9 parts of copper.

Set XCVIII.

(1) $\frac{5}{8}$. (2) 3s. (3) $3\frac{1}{8}$; $1\frac{1}{24}$; $1\frac{9}{16}$.
(4) 2d.; 4d.; 3d. (5) $16\frac{2}{3}$. (6) 3d.
(7) $7\frac{1}{2}$. (8) 2d. (9) 11820 yd. per min.
(10) $19\frac{5}{9}$. (11) $7\frac{1}{2}$. (12) 162.
(13) $22\frac{13}{16}$. (14) 731.496... metres per min.
(15) 4.1314... yd. per sec. (16) 19 nearly.
(17) $\frac{5}{848}$ centig. deg. per sec.
(18) 3s. $0\frac{3}{4}$d. nearly per ton per mile.
(19) 81.432 Fah. deg. per min.

(20) 273736.2... kilogram-metres per hour.
(21) 423 5... kilogram-metres per kilogram-degree.

Set XCIX.

(1) £7 7s.	(2) £9 8s.	(3) £5 18s. $10\frac{4}{5}$d.
(4) £15 12s. $10\frac{4}{5}$d.	(5) £37 10s. $10\frac{1}{2}$d.	(6) £28 10s. 9d.
(7) £428 9s. $0\frac{3}{4}$d.	(8) £34 14s. $11\frac{1}{16}$d.	(9) £72 4s. $2\frac{5}{8}$d.
(10) £31 5s. $5\frac{5}{8}$d.	(11) £46 10s. $11\frac{1}{4}$d.	(12) £61 3s. 1d.
(13) £23 5s. $0\frac{3}{8}$d.	(14) 16s. 6d.	(15) £11 5s. $8\frac{7}{16}$d.
(16) £2 2s. $6\frac{17}{40}$d.	(17) £2 7s. $10\frac{73}{320}$d.	(18) 8s. $9\frac{3}{10}$d.
(19) £17 3s. $4\frac{1}{5}$d.	(20) £44 12s. 6d.	(21) £117 16s. $3\frac{39}{320}$d.
(22) £50 1s. $11\frac{25}{32}$d.	(23) £4 12s. $11\frac{413}{2800}$d.	(24) £1 9s. $2\frac{1123}{1168}$d.
(25) £1 17s. $6\frac{27}{128}$d.	(26) £42 4s. $9\frac{3}{40}$d.	(27) £90 11s. $7\frac{241}{365}$d.
(28) £1 12s. $2\frac{1469}{10400}$d.	(29) 3.38... fr.	(30) £19 0s. 5.3...d.*
(31) £269 11s. $4\frac{8}{73}$d.	(32) £351 19s. $7\frac{38}{73}$d.	(33) £23 3s. $2\frac{62}{365}$d.
(34) £861 10s. $0\frac{681}{14600}$d.	(35) £435 18s. $1\frac{541}{18250}$d.	

Set C.

(1) 12 years.	(2) $\frac{69}{474}$.	(3) £1020.
(4) $1\frac{1}{2}$.	(5) 4 years.	(6) £380 10s.
(7) $2\frac{3}{4}$ years.	(8) $2\frac{1}{8}$.	(9) 3 years.
(10) £542 10s. $9\frac{3}{8}$d.	(11) £730.	(12) $17\frac{3}{10}$ days.
(13) 40 years.	(14) $\frac{1}{80}$.	(15) 60.
(16) $1\frac{13}{362}$.	(17) 30th June.	(18) $5\frac{242816}{18087987}$.
(19) £66.	(20) £456 17s. 6d.	(21) £219 7s. $4\frac{29}{37}$d.
(22) £2190.	(23) £750 10s.	(24) $2\frac{247}{308}$.
(25) 4.	(26) £291 12s. $1\frac{4}{7}$d.	(27) £45 2s. $9\frac{3}{4}$d.
(28) 25 years.	(29) £12 7s.	(30) £8 10s. $4\frac{1}{4}$d.
(31) £711 19s. $10\frac{1}{2}$d.	(32) 2.	(33) £2673 2s. 6d.
(34) £130.	(35) £332 12s. $7\frac{19}{28}$d.	

Set CI.

(1) £819 9s. 9d.	(2) £859 19s.	(3) £2401 3s. 0.48d.
(4) £578 16s. 3d.	(5) £8933 19s. 5.025d.	(6) £6810 19s. 6.72d.
(7) £557 4s. 5.5...d.	(8) £9671 2s. 3.5...d.	(9) £64 13s. 10.1...d.
(10) £921 11s. 10.0...d.	(11) £20 10s. 8.9...d.	(12) £25 3s. 10.6...d.
(13) £1 10s. 6.8...d.	(14) £36 5s. 0.8...d.	(15) £13 17s. 3.6...d.
(16) £444 6s. 6.5...d.	(17) £1029 18s. 2.7...d.	(18) £872 9s. 5.4...d.
(19) £669 2s. 7.7...d.	(20) £571 17s. 7.3...d.	(21) £824 12s. 8.1...d.
(22) 2s. 0.0...d.	(23) £13 11s. 11.0...d.	(24) £5 8s. 9.6...d.

* Reckoning a month the twelfth part of a year, and a day the thirtieth part of a month (see page 224).

(25) £1 2s. 4.9...d., if the rate be 5 per cent. (26) £2785 19s. 3.0...d.
(27) £3 0s. 4.8...d. (28) £486 19s. 4.7...d. (29) £1692 9s. 0.6...d.
(30) £4322 14s. 7.1...d.

Set CII.

(1) £1000. (2) £200000. (3) £462 1s. 4.5...d.
(4) £14500. (5) £824.
(6) £2000 ready money, for £2000 in three years would beco[illegible] £2185.454.
(7) £1350, very nearly. (8) £6592 2s. 8d. nearly.
(9) £4085 19s. $0\frac{12}{21}$d. (10) £709 10s. 8.3...d.
(11) £1474 1s. 11.3...d. (12) 3 years, very nearly.

Set CIII.

(1) £560 1[illegible]d.; £56 1s. 9d. (2) £920; £69.
(3) £1170; [illegible]6 1s. (4) £1280; £54 8s.
(5) £463 2s. [illegible]d.; £37 1s. (6) £132; £13 4s.
(7) £681 6s. 8d.; £43 16s.
(8) £2130 6s. 10.2...d; £149 2s. 5.7...d.
(9) £1700; £53 2s. 6d.
(10) £835 8s. 10.3...d.; £11 8s. 10.6...d.
(11) £444 8s. 3d. (12) £93 0s. $1\frac{1}{2}$d. (13) £56 5s. 9d.
(14) £30 16s. $10\frac{1}{2}$d. (15) £250 3s. 9d. (16) £61 3s. 1d.
(17) £20 5s. $1\frac{1}{2}$d. (18) £90 5s. $7\frac{1}{2}$d. (19) £86 2s. 4.0...d.
(20) £171 6s. $6\frac{3}{4}$d. (21) £9 1s. 5.4...d. (22) £21 2s. 10.5...d.
(23) £600; £94 11s. 6d. (24) £875; £71 8s.
(25) £708 4s. 5.7...d.; £54 8s. 0.2...d.
(26) £857 0s. 0.5...d.; £23 10s. 7.4...d.
(27) £53 8s. (28) £1825. (29) £1460.
(30) £21 6s. 8d. (31) £730. (32) £3 6s. 11.0...d.
(33) £1095. (34) £11 7s. 11.3...d. (35) £137 16s. 7.8...d.
(36) £574 13s. 8.3...d. (37) £32000 2s. 0.4...d. (38) £393 3s. 2.7...d.
(39) £672 19s. 1.8...d. (40) £8 16s. 0.6...d. (41) £5 3s. 3.7...d.
(42) £3770 7s. 8.0...d. (43) £4 9s. 8.1...d. (44) £66 9s. 9.4...d.
(45) 2.4...d. (46) £6 1s. 4.6...d. (47) £1231 0s. 8.1...d.
(48) £2 14s. 2d. (49) £38; £39 6s. 8d.; £39 6s. 10.6...d.
(50) £543 14s. 1.3...d.; 4.63... mo. (51) 7.79... mo.
(52) 4.51... mo. (53) £371 14s. 2.3...d. (54) £69 7s. 6.5...d.

Set CIV.

(1) £54 12s. (2) £4500. (3) £159 ~~8s. 6d.~~
(4) £6277 10s. (5) £16621 17s. 6d. (6) £24000.

(7) £6959 7s. 6d. (8) £7832 3s. $4\frac{40}{143}$d. (9) £3628 13s. 6d.
(10) £32 5s. 9d. (11) £550. (12) £200.
(13) $3\frac{9}{37}$. (14) £169 16s. $2\frac{88}{93}$d. (15) £2500; £3 2s. 6d.
(16) £$88\frac{7}{8}$. (17) They are equally good.
(18) £490 15s. (19) The 4 per cents. (20) Total cost=£9998.
(21) £5000. (22) £3120 16s. 8d. (23) $6\frac{13}{24}$.
(24) $33\frac{1}{3}$. (25) £395200000. (26) £20 12s.
(27) £3550. (28) £4540. (29) £591 11s. $6\frac{114}{187}$d.
(30) $20\frac{40}{108}$. (31) £112 10s. (32) £24 8s.
(33) £36. (34) £292 5s. $3\frac{128}{189}$d.
(35) It is increased by £101 5s. $10\frac{10}{17}$d. (36) £98175000.
(37) $2\frac{9}{10}$ per cent. (38) £52 10s. (39) By £2 0s. $3\frac{1888}{2003}$d.
(40) £6 15s. (41) £112 16s. (42) £25.
(43) $3\frac{1789}{2997}$. (44) 20.
(45) £22557355 14s. $4\frac{4}{5}$d. (46) An increase of £20 2s. $7\frac{11}{19}$d.
(47) £52 10s. more. (48) £168. (49) £60; £4000.
(50) £4250. (51) $52\frac{1}{2}$. (52) £2 18s. $9\frac{15}{17}$d.
(53) $98\frac{1}{3}$. (54) £250000 stock. (55) $4\frac{1}{20}$d.
(56) It depends on the price originally paid for Midland stock; if it were bought at par the answer would be $1\frac{63}{200}$.
(57) £272 13s. 3.6...d. (58) £8617 10s. (59) £3078.
(60) £4572 11s. 6d.

Set CV.

(1) £216 19s. 3.0...d. (2) £183 1s. 3d. (3) £94 7s. 3.5...d.
(4) £55 11s. 7.6...d. (5) £147 10s. 1.1...d. (6) £152 9s. 11.0...d.
(7) £432 7s. 10.6...d. (8) £28 17s. 9.8...d. (9) £26 2s. 10.2...d.
(10) £33 12s. 11.4...d.
(11) 3824.68... francs, 2215.72... rupees, 1297.56... roubles.
(12) 1047.37... dollars, 10235.15... marks, 345645.9... reis.
(13) 1710.035 gulden, 1439.07... guilders, 3094.92... lire.
(14) £6 4s. 0.4...d. (15) £93 10s. 6.4...d. (16) 2828.06... dollars.
(17) 22.4. (18) 9064.84... gulden. (19) £2138 1s. 4.0...d.
(20) 3560.47... florins. (21) 15.90... florins less.
(22) .99... (23) 4.86...
(24) The former; by it a debt of 100 guilders would produce 218.18... francs, by the other only 212.5 francs.
(25) 1233.65... dollars. (26) The latter, by 215.48... francs.
(27) 4.88... (28) 5548.06... francs. (29) 9193.30... marks.
(30) 114.91... (31) 184016.39... francs.
(32) A loss to A of £55 8s. 10.9...d.
(33) Genoa. To pay 100 francs in Paris there would be required in the different cases £3.93..., £3.913..., £3.912..., £3.8... respectively.
(34) 5.18...

(35) Geneva. Through Geneva 100 r. would cost £13.56..., through Paris £13.57.

(36) 25.22...

(37) The first; to pay 100 gulden in Vienna there would be required in the ways mentioned £7.04..., £7.05..., £9.4..., £9.6... respectively.

Set CVI.

(1) A, £480; B, £80; C, £40. (2) $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{16}$, $\frac{1}{8}$ respectively.

(3) £100, £66 13s. 4d., £33 6s. 8d. (4) £825, £495, £330.

(5) £21, £10 10s., £8 8s. (6) 40, 35, 25.

(7) £180 14s. 11d., £232 7s. 9d. (8) 2, 8, 18, 32.

(9) £705 13s. 4d., £529 5s., £882 1s. 8d.

(10) 19 cwt. 1 qr., 2 cwt. 1 qr., 3 cwt. 2 qr.

(11) 24, 12, 8, 6. (12) $28\frac{4}{5}$, $7\frac{1}{5}$, $3\frac{1}{5}$, $1\frac{4}{5}$.

(13) $28\frac{4}{5}$, $7\frac{1}{5}$, $3\frac{1}{5}$, $1\frac{4}{5}$. (14) 121, 117, 95, 32.

(15) 2 ac. 1 ro. 39 sq. po., 3 ac. 1 ro. 36 sq. po., 7 ac. 3 ro. 2 sq. po.

(16) £70 16s. 8d., £157 3s. 9d., £195 1s. $6\frac{3}{4}$d., £315 19s. $3\frac{1}{4}$d.

(17) 3s. $8\frac{8}{17}$d., 2s. $5\frac{11}{17}$d., $9\frac{16}{17}$d. (18) £7 7s., £13 2s. 6d., £8 18s. 6d.

(19) 4 ft., 4 ft., 2 ft., $1\frac{1}{3}$ ft. (20) £246 18s., £123 9s., £82 6s.

(21) £642, £856. (22) 120, 60, 30.

(23) £1076 3s. $9\frac{6}{7}$d. (24) Boy's share, 2; girl's, 4.

(25) £348 18s. $9\frac{16}{37}$d., £279 3s. $0\frac{12}{37}$d., £232 12s. $6\frac{10}{37}$d.

(26) £369 4s. $10\frac{1846}{2396}$d., £215 9s. $6\frac{1458}{2396}$d., £35 5s. $7\frac{1488}{2396}$d.

(27) 8s. 4d., 5s.

(28) £182, £91, £60 13s. 4d., £364, £121 6s. 8d.

(29) £9 9s., £14 14s., £6 6s. (30) 15s. $1\frac{1}{2}$d.

(31) 2s. 6d., 1s. 3d., 3s. 9d. (32) $\frac{1}{78}$, $\frac{2}{95}$, $\frac{1}{38}$.

(33) A man's share, $8\frac{1}{3}$; a woman's, 5; a child's, $3\frac{1}{3}$.

(34) £10 5s. $9\frac{447}{1837}$d., £9 19s. $10\frac{1274}{1837}$d., £9 14s. $4\frac{116}{1837}$d.

(35) .01 in., .015 in., .0225 in.

(36) £229 14s. $1\frac{1}{2}$d., £153 2s. 9d., £229 14s. $1\frac{1}{2}$d.

(37) £580 12s. $10\frac{26}{31}$d., £619 7s. $1\frac{5}{31}$d.

(38) £383 16s., £287 17s., £35 7s.

(39) 432, 324, 216. (40) $\frac{26}{47}$, $\frac{15}{47}$, $\frac{6}{47}$.

Set CVII.

(1) 276 sq. ft. 28 sq. in. (2) 6 sq. yd. 5 sq. ft. 79 sq. in.

(3) $197\frac{22}{27}$ sq. ft. (4) 80.0475 sq. in. (5) $217\frac{13}{24}$ sq. yd.

(6) 213 ac. 3 ro. 25 sq. po. (7) 8 ac. 2 ro. 15 sq. po. 29 sq. yd.

(8) 4 ft. 5 in. (9) $8\frac{49}{72}$ sq. ft. (10) $453\frac{29}{32}$ sq. yd.

(11) 3 ft. (12) 4 chains. (13) 8.

(14) 1 ac. 1 ro. 25.91... sq. po. (15) 1 ft. 3 in.

(16) 51 sq. in. (17) 1196.94... sq. yd. (18) $659\frac{1}{8}$ sq. ft.
(19) 10.196875 sq. in. (20) $30\frac{7}{8}$ sq. in. (21) 33454080 yd.
(22) 2772.61... sq. ft. (23) 36 yd. (24) 1760 miles.
(25) 32 yd. (26) $665\frac{1}{2}$ sq. ft. (27) $129\frac{48}{81}$ sq. yd.
(28) 30.9... sq. ft. (29) 4 ac. 1 ro. 27 sq. po. $13\frac{1}{4}$ sq. yd.
(30) $1022\frac{11}{12}$ sq. ft. (31) $\frac{871}{1221}$. (32) $\frac{1}{2}$ in. to the mile.
(33) 6720. (34) 6 in. (35) $24\frac{81}{96}$ sq. ft.
(36) £1 10s. $7\frac{1}{2}$d. (37) 29376. (38) 85 sq. in.
(39) $2\frac{1}{7}$ in. (40) 82 sq. ft. $5\frac{1}{2}$ sq. in. (41) 15 ft.
(42) 15 ft. 9 in. (43) 2.5 in.

SET CVIII.

(1) 1 cub. ft. 1485 cub. in. (2) 3 cub. ft. 1566 cub. in.
(3) 20 cub. ft. 1260 cub. in.
(4) 156 cub. yd. 25 cub. ft. 864 cub. in.
(5) 5 cub. ft. 1566 cub. in. (6) 1728.
(7) .001953125 cub. in. (8) $2\frac{1}{4}$ ft. (9) $140\frac{5}{8}$ cub. ft.
(10) 101. (11) 16349.04. (12) $\frac{1}{128}$ cub. in.
(13) 14 ft. $2\frac{2}{3}$ in. (14) 3888.
(15) 259332805349.9, taking $\pi = 3.141592653589$.
(16) 1728000. (17) 78. (18) .001022...cub. in.
(19) 210. (20) $7\frac{41}{768}$. (21) 130680.
(22) 13824. (23) 14.49... in. (24) 86.5...
(25) .064325390625 cub. in. (26) 16.29...
(27) 57.93... (28) 65.44... cub. in. (29) 449.28... sq. ft.
(30) .003003001 cub. in. (31) 3811.2... cub. in.
(32) $1\frac{23}{64}$ ft. (33) 2205.
(34) 24 cub. ft. $1330\frac{1}{2}$ cub. in. (35) 72.4... inches.
(36) 2.09... in. (37) $3\frac{5}{8}$ in. (38) 25500.
(39) .9027027. (40) 13089.9...

SET CIX.

(1) 20, 60, 80, 100. (2) 300, 1000, 500. (3) 40000, 9000.
(4) .5, .05, .1, .04. (5) .009, .03, .001. (6) .0006, .0002.
(7) $\frac{1}{2}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$. (8) $\frac{4}{5}$, $\frac{5}{8}$, $\frac{4}{9}$, $\frac{8}{9}$. (9) $\frac{7}{10}$, $\frac{3}{40}$, $\frac{1}{200}$, $\frac{9}{70}$.
(10) $1\frac{1}{2}$, $2\frac{1}{2}$, $1\frac{1}{3}$, $1\frac{2}{3}$. (11) $1\frac{1}{4}$, $2\frac{1}{4}$, $1\frac{4}{5}$, $1\frac{2}{7}$. (12) $3\frac{1}{3}$, $6\frac{2}{3}$, $5\frac{6}{7}$, $5\frac{5}{9}$.
(13) 18, 25, 19. (14) 22, 31, 101.
(15) Between 100 and 1000; between 290 and 300.

SET CX.

(1) 23. (2) 34. (3) 44. (4) 46.
(5) 53. (6) 65. (7) 91. (8) 71.
(9) 59. (10) 49. (11) 88. (12) 97.

(13) $\frac{19}{48}$. (14) $\frac{2}{3}$. (15) $\frac{29}{35}$. (16) $\frac{47}{99}$.
(17) $1\frac{11}{31}$. (18) $1\frac{16}{61}$. (19) $1\frac{43}{62}$. (20) $8\frac{3}{10}$.
(21) 4.1. (22) 2.8. (23) .079. (24) .0092.

Set CXI.

(1) 712. (2) 916. (3) 943. (4) 565.
(5) 674. (6) 786. (7) 888. (8) 999.
(9) 909. (10) 1148. (11) 1247. (12) 1346.
(13) 2593. (14) 3295. (15) 3075. (16) 3892.
(17) 3987. (18) 7008. (19) 7897. (20) 7989.
(21) 8798. (22) 9080. (23) 9930. (24) 80080.
(25) 26253. (26) 24753. (27) 48795. (28) 51.86.
(29) 729.6. (30) 25.653. (31) .2951. (32) .0759.
(33) .0298. (34) $2\frac{983}{1010}$. (35) $2\frac{436}{989}$. (36) $2\frac{813}{3446}$.
(37) 9007000. (38) 1040400. (39) .000400503. (40) .0090990089.

Set CXII.

(1) 2.236... (2) 4.123... (3) 4.242... (4) 10.049...
(5) 8.185... (6) 14.730... (7) 18.601... (8) 20.099...
(9) 35.057... (10) 37.509... (11) 37.907... (12) 46.010...
(13) 5517801. (14) 46.16275...; 2116, 212521, 21307456, 2130930244, 213099487129, 21309994875625.
(15) .400124... (16) .283019... (17) 4.009987... (18) .050309...
(19) 1.710263... (20) 5.408326... (21) .036055... (22) .011401...
(23) .433012... (24) .832050... (25) .774596... (26) .845154...
(27) 1.788854... (28) 2.478478... (29) 2.943920... (30) 10.085993...
(31) 3.9999..., the limit being 4. (32) 0.
(33) 1.7737... (34) .1010... (35) 1.7782... (36) 1.9615...

Set CXIII.

(1) 10, 20, 40. (2) 80, 2000. (3) 300, 1000.
(4) .1, .3, .07. (5) .01, .004. (6) .002, .006.
(7) 12 or 13. (8) 8 + 9 + 10, or 8 + 9 + 10 − 1, or 8 + 9 + 10 − 2.
(9) 22. (10) 32. (11) 45. (12) 64.
(13) 56. (14) 86. (15) 78. (16) 99.
(17) .55. (18) 7.6. (19) .29. (20) .18.
(21) 234. (22) 446. (23) 674. (24) 883.
(25) 979. (26) 1172. (27) .0679. (28) .0387.
(29) .0209. (30) 150.3. (31) 208.7. (32) 60.08.
(33) .6509. (34) 778.7. (35) 9.598. (36) 1.442...
(37) 1.817... (38) 3.107... (39) 3.608... (40) 4.805...
(41) 5.788... (42) 5.990... (43) 8.900... (44) 13.186...

(45) 16.779.. (46) 17.906... (47) 20.907... (48) .935228...
(49) .360882... (50) .208008... (51) .070405... (52) .216166...
(53) .426859... (54) 1.062658... (55) 2.971961... (56) .747...
(57) 5.768... (58) .403... (59) 4.999... or 5. (60) 5.848...
(61) .770...

SET CXIV.

(1) .141... (2) .447... (3) .251... (4) 1.406...
(5) $\sqrt[5]{32}$, $\sqrt[6]{729}$, $\sqrt[6]{64}$, which = 2, 3, 2 respectively.
(6) 510. (7) .047. (8) 2.56... (9) 1.16...
(10) 1.09... (11) 1.65... (12) 1.7. (13) .19.
(14) 625 links. (15) 1 ft. $1\frac{1}{2}$ in. (16) The 16th, 18th, and 24th; thus, $\sqrt[16]{5}=\sqrt[2]{\sqrt[2]{\sqrt[2]{\sqrt[2]{5}}}}$, $\sqrt[18]{5}=\sqrt[2]{\sqrt[3]{\sqrt[3]{5}}}=\ldots\ldots$, $\sqrt[24]{5}=\sqrt[2]{\sqrt[2]{\sqrt[2]{\sqrt[3]{5}}}}=\ldots\ldots$
(17) $4\frac{1}{4}$. (18) 14 ft. 9.08... in. (19) 1 ft. 8.34... in.
(20) 5.55 in. (21) .012. (22) 5.
(23) 7 ft. $5\frac{4}{5}$ in. (24) 336.973... (25) 43.7... centim.
(26) 1.29... in. (27) 4.

SET CXV.

(1) $\frac{1}{990}$, $\frac{1}{1980}$.

(2) Incommensurable number; because all surds are incommensurable numbers, while there are incommensurable numbers which are not surds.

(3) $\sqrt{2}$ and $\sqrt{18}$, the former being contained three times in the latter.

THE END.

PRINTED BY VIRTUE AND CO., LIMITED, CITY ROAD, LONDON.